

## On the ionization in non-isothermic atmospheres, by *A. Pannekoek*.

In *Harvard Bulletin* 841 the formula derived in *B. A. N.* 110 for the ionization in atmospheres that are not in thermodynamic equilibrium, is criticized by Prof. B. GERASIMOVIC<sup>v</sup>. His first objection is that the conditions in such an atmosphere are too complicated to be treated in this way by simple deductions. Of course he is right that the simple suppositions, from which the formula was deduced, cannot be strictly true. But they will be nearer to the truth than the first approximation of the supposition of thermodynamic equilibrium. When we consider the valuable and important results that have been obtained with SAHA's formula, which strictly holds only for isothermal gases, there is no reason to reject a second approximation because it does not exactly represent the complicated conditions in a stellar atmosphere.

If we try to evaluate these conditions we must consider the different forms of energy exchange going on: (*A*) by absorbing a radiation quantum  $h\nu$  the atom converts it into quantum energy; (*A'*) by returning to the first state this quantum is radiated by the atom. (*B*) An electron colliding with an atom gains kinetic energy by absorbing radiation; or (*B'*) it loses kinetic energy, the difference being radiated away. (*C*) By absorbing a radiation quantum  $h\nu = \chi_0 + \frac{1}{2}mv^2$  the atom is ionized and the electron gets a velocity  $v$ ; (*C'*) an electron with velocity  $v$  is captured and  $h\nu = \chi_0 + \frac{1}{2}mv^2$  is radiated away. (*D*) By an inelastic collision of an electron with an atom kinetic energy  $h\nu$  is converted into quantum energy; (*D'*) by a hyperelastic collision quantum energy is converted into kinetic energy. (*E*) A colliding electron ionizes an atom; (*E'*) in a triple collision an electron is captured. In thermodynamic equilibrium each of these processes is balanced by its converse process.

In the cases we have to consider here, the first source of the energy is radiation falling from one side upon the atoms. We assume as an exact rule that the action of radiation on an atom is independent of the states of other atoms (thus of statistical parameters  $T$  and  $p$ ) and independent of the direction from which the radiation is coming. Thus in the case of black radiation of temperature  $T_1$ , coming from directions filling a fraction  $\beta$  of the surrounding sphere, the exciting and ionizing action on the atoms is  $\beta \times$  these actions in the case of isotropic radiation of the same temperature  $T_1$ . Compared with this case the processes *A*, *B*, *C* are weakened. Considering at first only *A* and *A'*, then between two quantum states of energy  $\chi_r$  and  $\chi_s$ , if they were the only ones existing, the relation  $n_s : n_r = \beta e^{-(\chi_r - \chi_s)/kT_1}$  would hold; since such relations cannot exist at the same time between

all quantum states, a continual adjustment must take place by cyclic processes like 1-3-2-1 occurring more often than the reverse cyclus 1-2-3-1. As a result we will have for the relative number of two consecutive states  $n_{r+1} : n_r = \gamma e^{-(\chi_r - \chi_{r+1})/kT_1}$ , where the coefficient  $\gamma$  is somewhat larger than  $\beta$  and varies with  $r$ . As soon as physical theory provides for the transition probabilities belonging to each pair of quantum states the resulting distribution may be computed. It is not identical with the distribution belonging to some other temperature; but we may indicate some middle temperature, which may be called the inner temperature of the atoms, lower than  $T_1$ . By the processes *C* and *C'* the liberated electrons get a mean energy of translation, which, after EDDINGTON's computations, immediately after expulsion, though not Maxwellian, corresponds to a temperature somewhat below  $\frac{2}{3}T_1$ , but gradually rises to  $T_1$ , just as in the isothermal case. Only the rate of ionization is lower, and the rate of transformation of radiation into this high velocity kinetic energy is  $\beta$  times less. By the processes *B'* and *D* this kinetic energy is transformed into radiation emitted and into quantum energy; their rate is determined by factors not containing  $\beta$ , thus the mean kinetic energy of the electrons is lowered and a stationary sequence of energy exchanges radiation—kinetic electron energy—(radiation or) quantum energy—radiation takes place, not balanced by the converse processes. The process *B* contains  $\beta$  and thus cannot balance *B'*. The balance of the collisions *D* and *D'*, in consequence of the result of KLEIN and ROSSELAND, that for the same quantum transition the probability coefficients are reciprocal to the initial energies, will be such that the electrons gain or lose if their translation temperature is lower or higher than the inner temperature of the atoms. Thus in the resulting equilibrium the temperature measured by the mean kinetic energy of the electrons is lower than  $T_1$ , but higher than the temperature measured by the distribution of quantum states. More precise statements can only be made in consequence of a more precise knowledge of the probability coefficients of these transitions. In a mixture of gases the electrons are exchanging energy with all the different kinds of atoms.

In deriving the ionization in such an atmosphere we may, after FOWLER's computations, neglect the ionization by collisions (*E* and *E'*). Then only two factors have to be considered: the ionizing power of radiation and the rate of electron capture. The radiation falling on the atoms is certainly not a simple black radiation; it is composed of radiations coming from

all directions of the lower hemisphere; each element, by the atomic emissions and absorptions, is not black, and if approximately it may be assumed to be black. its temperature and intensity vary with the direction. For the character and the variation of these radiations we may use the empirical data of the solar radiation, as it varies from the centre to the border of the disc. With the same order of error, which we commit in substituting for this variable radiation a mean solar radiation and take it a black radiation of the effective temperature  $T_x$ , we may assume in our computations the SCHWARZSCHILD approximation of a black radiation of temperature  $T_x$  with  $\beta = 1/2$ . Further approximations appear possible if there is a need for them. In the computation of the captures the uncertainty is greater. In the denominator integral of WOLTJER's formula a Maxwellian distribution of velocities is assumed; though this is not exactly right, the deviations certainly will not be important. More difficulties are presented by the temperature of the electron gas, because it cannot be computed directly. In the elementary theory equilibrium of emission and absorption of black radiation is assumed; thus we find  $T = T_x \beta^{1/4}$ . If we suppose that in the quantum distribution only the lowest quantum states are relevant, the middle temperature introduced above as inner temperature of the atom is determined by  $\gamma e^{-h\nu/kT_x} = e^{-h\nu/kT}$ , or  $\gamma I(\nu, T_x) = I(\nu, T)$ , where  $\gamma$  is slightly larger than  $\beta$ . If now the atmosphere consists of a mixture of many gases, whose resonance lines are rather regularly distributed over the whole spectrum, we have instead of this formula  $\sum \gamma I(\nu, T_x) = \sum I(\nu, T)$ . For the sum total of these monochromatic radiations, if only regularly distributed, we may take approximately the black radiation; thus the mean inner temperature of these atoms will deviate not very much from the temperature  $T_x \gamma^{1/4}$ . Since  $\gamma$  somewhat exceeds  $\beta$ , and the kinetic temperature of the electrons is still higher, the temperature of the electron gas used in the ionization formula would be better assumed somewhat higher than the value  $T$  that has been adopted in *B. A. N.* 110. The qualitative conclusions as to the deviations from SAHA's formula will hardly need any correction in this case. In the application of the formula to the terrestrial atmosphere (*Proc. Amsterdam Acad.* 29, 1165) the temperature of the gas has been determined by other considerations.

A second objection against my results is raised because the coefficient  $\psi(\nu)$  was taken proportional to  $\nu^{-3}$  in stead of proportional to  $\nu^{-4}$ . I have already remarked in my article that by the rapid decrease of  $I(\nu)$  with increasing  $\nu$  in all practical cases an

error in this exponent would have little influence on the result. If in the integrals the value  $\nu^{-4}$  is introduced for  $\psi(\nu)$ , we get

$$\int_{\nu_0}^{\infty} \nu^{-4} \frac{4\pi h\nu^3}{c^2} e^{-h\nu/kT_x} d\nu = \frac{4\pi h}{c^2} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu} e^{-h\nu/kT_x} = \frac{4\pi h}{c^2} Ei\left(-\frac{h\nu_0}{kT_x}\right)$$

$$\int_{\nu_0}^{\infty} \nu^{-4} \frac{8\pi h\nu^3}{c^2} e^{-h\nu/kT} d\nu = \frac{8\pi h}{c^2} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu} e^{-h\nu/kT} = \frac{4\pi h}{c^2} Ei\left(-\frac{h\nu_0}{kT}\right).$$

The factor by which SAHA's expression has to be multiplied thus becomes  $\frac{1}{2} Ei\left(-\frac{h\nu_0}{kT_x}\right) : Ei\left(-\frac{h\nu_0}{kT}\right)$ .

If we put  $Ei(-x) = -\frac{1}{x} e^{-x} Q(x)$ , then our formula in *B. A. N.* 110 has still to be multiplied with  $Q(h\nu_0/kT_x) : Q(h\nu_0/kT)$ . For values of  $x$  below  $x = 10$  this function  $Q$  has been computed, as given in the following table, by means of the tables of  $Ei(x)$  given by BRETSCHEIDER (*Schlömilch Zeitschr.* 6, 132, 1861). For larger  $x$  we may make use of the semiconvergent series (N. NIELSEN, *Theorie des Integrallogarithmus* p. 37)

$$Ei(-x) = e^{-x} \left( -\frac{1}{x} + \frac{1}{x^2} - \frac{2!}{x^3} + \frac{3!}{x^4} - \dots \right) + R_n; R_n < \frac{n!}{x_{n+1}}$$

$$\text{or } Q(x) = 1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \dots$$

$x$	$Ei(-x)$	$Q(x)$	$x$	$Ei(-x)$	$Q(x)$
1.	-0.21938	0.5963	7.	-0.1155.10 <sup>-3</sup>	0.8865
2.	-0.04890	0.7226	8.	-0.3767.10 <sup>-4</sup>	0.8982
3.	-0.01305	0.7863	9.	-0.1245.10 <sup>-4</sup>	0.9078
4.	-0.003779	0.8254	10.	-0.4157.10 <sup>-5</sup>	0.9156
5.	-0.001148	0.8521	20.		0.9544
6.	-0.360.10 <sup>-3</sup>	0.8716	100.		0.9902

For the practical cases where  $h\nu/kT$  is large, these factors are not much below 1. Thus e.g. for  $h\nu = 5.12$  Volt and  $T_x = 6000^\circ$   $h\nu/kT_x = 9.9$ ,  $h\nu/kT$  (for  $T = T_x \psi^{1/2}$ ) = 11.8, and the quotient of the values  $Q$  becomes 0.985. When the formula is applied to the ionization of nitrogen, oxygen and hydrogen in the terrestrial atmosphere we have  $h\nu = 13-15$  Volts,  $h\nu/kT_x = 28$ ,  $h\nu/kT = 700$ ; here the correction factor is nearly 0.97. When we consider that in the computation of ionization factors of order zero are often neglected and only the powers of 10 are material, it is clear that the errors of our formula from this source are insignificant.