BULLETIN OF THE ASTRONOMICAL INSTITUTES OF THE NETHERLANDS.

1937 May 8. Volume VIII. No. 297.

COMMUNICATION FROM THE ASTRONOMICAL INSTITUTE AT AMSTERDAM.

Limb Darkening in the Eclipsing Variable 21 Cassiopeiae,

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1. In discussions of photometric measures of eclipsing variables the aim of the computers is usually to derive a set of elements and to show that the measures can be satisfactorily represented by these elements. Because the amount of limb darkening of the stellar discs was unknown, H. N. RUSSELL and HARLOW SHAPLEY in their fundamental discussion of the problem constructed tables for two extreme suppositions, viz. uniform discs and complete darkening at the limb. A comparison of the "uniform" (U) and the "darkened" (D) solutions could indicate the influence of our uncertainty about the value of this datum.

The recent development of our astrophysical knowledge about stellar atmospheres has shown the connection between the amount of limb darkening for different colors and the coefficients of continuous absorption. It is now possible to compute by theory the limb darkening for different wave lengths in stars of different spectral type. So limb darkening to day is an important astrophysical datum by which our theories about the stellar atmospheres may be tested. Besides our own sun we have only the eclipsing variables as possible sources of information about this datum; this indicates the importance of attempts to determine the amount of limb darkening from photometric measures of eclipsing variables. If this amount is denoted by x, it means that the surface intensity $I = I_0(1 - x + x \cos \gamma)$, where $\sin \gamma$ is the distance to the centre of the disc; for $x = 1$ the limb is completely darkened.

Some results about limb darkening have been obtained in the last years. Especially the introduction of the photoelectric cell in stellar photometry has improved the accuracy of intensity measures to such a degree that it does not seem impossible to derive at least for some stars reliable values for the coefficient x. It is then, however, not sufficient to find a value of x that satisfies the observations; we wish at the same time to be sure about the degree of certainty with which it is given by the observations, i.e. to derive the mean error of x. This was the chief object of the present investigation.

2. We assume different values of x, the coefficient of limb darkening, and for each of them we derive the best set of elements and compare it with the observations. The regular way in solving the problem of deriving the best elements and their mean errors would have been the same as is followed in least squares corrections of a planetary orbit: to compute differential quotients giving the dependence of each observation on small variations of the different elements. We applied the principle of a least squares solution in another way. If we take three (e.g. equidistant) values of an unknown, derive for each case the differences Obs.-Comp. and compute the sum total of their squares $\Sigma\varepsilon^2$, then – if the most probable value of the unknown is situated within the chosen limits – the three values of $\Sigma\varepsilon^2$ can be represented by a parabola. The most probable value of the unknown is then the abscissa belonging to the top of the parabola, the minimum of $\Sigma\varepsilon^2$.

The same parabola allows to find the mean error. In the simplest case of one unknown x, for which there are n observations of equal weight with a mean value x_0, we have $\Sigma\varepsilon^2 = \Sigma(x - x_0)^2$. For a different value $x_0 + \Delta$ the sum total of the error squares $\Sigma\varepsilon'^2 = \Sigma\varepsilon^2 + n\Delta^2$. If for Δ we take μ_x, the mean error of x, $\Sigma\varepsilon'^2 = \Sigma\varepsilon^2 + n\mu_x^2$. Since $\mu_x^2 = \Sigma\varepsilon^2/n(n - 1)$ we have $\Sigma\varepsilon'^2 = (n/n - 1)\Sigma\varepsilon^2$. Hence in the parabola representing the sum of error squares as a function of x we have not only to look for x_0, the abscissa of the minimum, but also for the abscissas for which the ordinate is $n/(n - 1)$ times this minimum; they are situated at distance μ_x to both sides of x_0.

In the case of more unknowns an analogous result can be derived. If the unknowns are denoted by x, y, z, we have equations of the form $ax + by + cz = l + \varepsilon$, where l is the observed quantity. The normal equations have the form $[aa]x + [ab]y + [ac]z = [al]$, etc., and after elimination of y and z, following the Gaussian algorithmus, we have $[aa2]x = [al2]$, where $[aa2]$ is the weight of x_0. If now in stead of the most probable x_0 another value $x_0 + \Delta$ is introduced, the change of $\Sigma\varepsilon^2$ is not simply $[aa]\Delta^2$, because by a change of x the other unknowns y and z too are changed. It is easy to take these changes duly into account and to show that the real change in $\Sigma\varepsilon^2$ amounts to $[aa2]\Delta^2$. Hence for a change of μ_x in x_0 we have

$$\Sigma\varepsilon'^2 = \Sigma\varepsilon^2 + [aa2]\mu_x^2 = \Sigma\varepsilon^2 \{1 + 1/(n-m)\} = = \frac{n-m+1}{n-m}\Sigma\varepsilon^2,$$

where n is the number of equations and m is the number of unknowns.

3. We suppose a circular orbit and circular discs for the stars. If r is the radius of the large star, kr the radius of the small star (unit is the radius of the orbit), i the inclination of the orbit, then d, the apparent distance of the centres may be expressed as a function of $\vartheta = 2\pi(t - t_0)/P$:

$$\left(\frac{d}{r}\right)^2 = \frac{\cos^2 i}{r^2} + \frac{\sin^2 i}{r^2}\sin^2\vartheta.$$

Then α, the fraction of the light of the small star that is occulted by the large star, may be computed as a function of k and d/r. For the case of uniform discs ($x = 0$) RUSSELL determined it as a function of k and p, where $d/r = 1 + kp$, and then, for the purpose of finding the elements from observations, reversed the function and gave $p = f(k, \alpha)$ in his Table I[1]). An extensive and accurate table for $\alpha = f(k, d/r)$ has been computed and published by Dr. MANFRED WEND [2]). The same table may be used if the small star occults part of the large star, because for the same values of k and d/r the same area is covered, which is the fraction αk^2 of the large star.

For the case of complete darkening of the limb ($x = 1$, $I = I_0 \cos\gamma$), SHAPLEY has computed analogous tables, for which $p = f(k, \alpha)$ is given in Tables Ix and Iy of RUSSELL's later paper.

In the case of limb darkening x the intensity $I_0(1 - x + x\cos\gamma)$ can be expressed for each point of the disc as a linear function of the "uniform" and the "darkened" intensities:

$$I(x) = (1 - x)\, I(0) + x\, I(1).$$

This holds also for the occulted part of the disc, provided it is expressed in all these cases in the same unit I_0. The values of α in the tables, however, are expressed in the total uneclipsed brightness; for the U-case this unit is $\pi r^2 I_0$, for the D-case it is $^2/_3\,\pi r^2 I_0$, for the general case it is $\pi r^2 I_0(1 - {}^1/_3\,x)$. Hence

$$\alpha(x) = \frac{(1-x)\,\alpha_U + {}^2/_3\,x\,\alpha_D}{1 - {}^1/_3\,x}\;{}^{1)}.$$

In this way from RUSSELL's tables other tables may be derived giving α as a function of k and d/r for different values of x.

4. The eclipsing variable YZ = 21 Cas was chosen as a first instance to apply this method of working. An extensive series of measures with the photoelectric cell has been made by C. M. HUFFER at Madison[2]). The star is an A 3 star with a period of $4^d{\cdot}4672$, which is almost exactly cut in two equal intervals by the two eclipses. From the normal magnitude 5·6 the star decreases $0{\cdot}41^m$ in the primary, $0{\cdot}07^m$ in the secondary minimum. The duration of the eclipses is 0·31 days; the secondary minimum has a constant phase of 0·10 days, indicating a total eclipse of the small star. The primary minimum, due to an annular eclipse of the large star, shows a continuous variation, indicating a considerable amount of limb darkening. For our computations the tables of "reflected normals"[3]) (each based on 4 observations) were used. They give the magnitude difference between 21 and 23 Cas; subtracting the mean difference outside the eclipses $0{\cdot}343^m$, we find for each normal the magnitude difference with the unobscured light, from which the decrease in brightness ΔI as a fraction of the total light could be computed. The results for the primary minimum are found in Table 2, column 2, and they are plotted in Fig. p. 147. The normals during the constant phase of the secondary minimum give a mean decrease of 0·0636 of the total light; hence the large star emits 0·9364 of the combined light. The maximum decrease in the midst of the primary minimum is 0·3085 of the total light; so the part of the light of the large star which is occulted at maximum phase $\alpha_0 = 0{\cdot}3085 : 0{\cdot}9364 = 0{\cdot}3295$.

In the case of a central eclipse for the maximum phase this value, for a given x, determines the ratio of the radii k. Putting $k = \sin\beta$ we find by an easy integration $\alpha_0 = \sin^2\beta$ for $x = 0$, $\alpha_0 = 1 - \cos^3\beta$ for $x = 1$, hence

$$0{\cdot}3295 = \frac{3-3x}{3-x}(1-\cos^2\beta) + \frac{2x}{3-x}(1-\cos^3\beta),$$

[1]) *Astrophys. J.*, **35**, p. 332.
[2]) MANFRED WEND, *Eine Tafel zur Theorie der Bedeckungsveränderlichen;* Diss. Leipzig 1931.

[1]) The statement in RUSSELL's paper *Aph. J.* **36**, p. 70 and 240, on the derivation of α (x) is not right.
[2]) *Publications of the Washburn Observatory,* Vol. **15**, part 2.
[3]) l. c. p. 115.

an equation of the 3[d] degree in $\cos\beta$, which can be solved for every x and then affords the corresponding k. In the general case of an annular eclipse, if the minimum distance of the centres at the maximum phase d_0/r is given, x again determines the ratio k. So besides x as the first variable, the minimum distance d_0/r and the ratio k together constitute one second independent variable. It is easier, then, to choose k and to compute d_0/r in the following way. From

$$0{\cdot}3295 = \frac{3 - 3x}{3 - x}\,\alpha_{0\mathrm{U}} + \frac{2x}{3 - x}\,\alpha_{0\mathrm{D}}$$

where $\alpha_{0\mathrm{U}} = k^2 = \sin^2\beta$ we find $\alpha_{0\mathrm{D}}$; furthermore its central value $\alpha_{0\mathrm{D}}(ce) = 1 - \cos^3\beta$, hence $\alpha_{0\mathrm{D}} : \alpha_{0\mathrm{D}}(ce)$ is known, and RUSSELL's Table Iy for the darkened solutions gives the corresponding d_0/r.

For the computation of the other phases we want the inclination i which is directly connected with d_0/r, because here $\vartheta = 0$. For the beginning and the end of the eclipse (anomaly $= \vartheta_b$) we have outer contact of the discs, hence $d/r = 1 + k$, and

$$(1 + k)^2 = \frac{\cos^2 i}{r^2} + \frac{\sin^2 i}{r^2}\sin^2\vartheta_b$$

$$\left(\frac{d_0}{r}\right)^2 = \frac{\cos^2 i}{r^2}, \text{ hence}$$

$$tg^2 i = \frac{1}{\sin^2\vartheta_b}\left(\frac{(1 + k)^2}{(d_0/r)^2} - 1\right); \left(\frac{d/r}{d_0/r}\right)^2 = 1 + tg^2 i \sin^2\vartheta .$$

When for given values of t and ϑ d/r has been found, then by using k and x the value of α and the obscured fraction of the total light is determined.

It is, however, necessary to consider $\sin\vartheta_b$ also as an unknown, since the moment of the beginning and the end of the eclipse can be read directly only with great uncertainty. If we add it as a third independent variable it will be determined from the entire light-curve. So three independent elements can be varied to adapt the theoretical light-curve to the observations: x, k (including d_0/r) and $\sin\vartheta_b$ – therewith considering $\alpha_0 = 0{\cdot}3295$ for the maximum phase in the primary minimum as an invariable given datum.

5. After for each of the values $x = 0{\cdot}40$, $0{\cdot}50$, $0{\cdot}60$, $0{\cdot}70$ solutions had been found that represented the observed curve well, the computations were made for adjacent different values of the other unknowns. For each x three values of k were chosen, and for each of these cases three values of $\sin\vartheta_b$ were assumed. For all these cases the diminution of brightness, in fraction of the total light, was computed for $\sin\vartheta =$ 0·035, ·056, ·070, ·105, ·140, ·175, corresponding to $t - t_0 =$ 0·025, ·04, ·05, ·075, ·10, ·125 days. This was sufficient to trace the curve exactly; its outer parts were fixed by the beginning $\sin\vartheta_b$ and the value for $\sin\vartheta = 0{\cdot}175$, taking into account that there the intensity decreases as $(\sin\vartheta_b - \sin\vartheta)^{3/2}$. Then for all the normal points the deviation O—C was derived (in units 0·001 of the full light, which nearly corresponds to $0{\cdot}001^{\mathrm{m}}$) and $\Sigma\varepsilon^2$ was computed.

In Table 1 the results are collected. Each line gives $\Sigma\varepsilon^2$ for the three assumed values of $\sin\vartheta_b$, as well as the number of permanencies and variations of sign.

TABLE 1.

Primary Minimum. Results for different hypotheses.

x	k	$\sin\vartheta_b$			$\Sigma\varepsilon^2$			Perm.-Var.			$\sin\vartheta_b$	$\Sigma\varepsilon^2$
0·40	0·5428	0·221	·213	·209	5206	1813	1404	34—7	26—11	24—13	0·2090	1404
	0·550	0·220	·216	·212	1309	935	1544	19—20	23—15	23—15	0·2165	928
	0·555	0·224	·220	·216	3238	2523	3020	30—11	26—13	26—9	0·2196	2474
0·50	0·534	0·217	·214	·210	2223	1377	1324	27—11	26—11	28—9	0·2118	1196
	0·540	0·220	·215	·210	1673	892	2453	22—19	21—15	27—11	0·2141	861
	0·545	0·224	·220	·217	1921	1614	1985	23—15	23—15	33—5	0·2205	1603
0·60	0·525	0·221	·216	·214	2038	985	1143	27—11	21—19	24—13	0·2163	981
	0·530	0·224	·220	·217	1902	937	869	24—15	18—19	18—19	0·2181	828
	0·535	0·224	·220	·217	1309	1154	2335	19—17	23—15	29—9	0·2217	978
0·70	0·5153	0·224	·219	·214	2556	865	1424	31—7	20—17	24—11	0·2178	794
	0·520	0·224	·222	·220	1444	1112	1068	24—14	15—19	25—12	0·2207	1051
	0·525	0·224	·220	·217	1404	1429	2258	22—15	27—11	30—7	0·2225	1262
0·40	0·5471	0·216	·213	·210	1000	884	1504	19—17	20—13	26—9	0·2140	828
0·50	0·5385	0·220	·215	·210	1539	834	1764	26—11	23—13	26—11	0·2153	830
0·60	0·5298	0·222	·219	·216	1556	814	837	25—13	16—19	21—13	0·2177	730
0·70	0·5153										0·2178	794
0·70 (·3105)	0·5174	0·220	·2175	·215	974	831	1042	24—15	19—17	19—17	0·2178	829

Representing the three values of $\Sigma\varepsilon^2$ by a formula $a + b\Delta + c\Delta^2$ we find the most probable value of $\sin \vartheta_b$ and the corresponding minimum of error squares in the last columns. They show the smallest $\Sigma\varepsilon^2$ attainable for each of the three values of k; treating them in the same way, we find the most probable k belonging to each x. For $x = 0·70$ the case is different; because the first value $k = 0·5153$ corresponds to $d/r = 0$, central eclipse, it is the lowest value that is possible; since a smaller value is not possible for $\Sigma\varepsilon^2$ this k has to be assumed. For each of the other three cases of corresponding x and k values the computation was repeated (in the lower part of Table 1) for different $\sin \vartheta_b$, to find the most probable value directly as well as the residual $\Sigma\varepsilon^2$. The object of this repetition was chiefly to see what differences in $\Sigma\varepsilon^2$ may be expected simply from the unavoidable small errors in the computations; it appears that they may amount to nearly 100. Whereas the parabolic determination of k in the first three cases had given for $\Sigma\varepsilon^2$ (min.) 667, 820, 828, the new derivation of $\sin \vartheta_b$ yielded 828, 830, 730. A new computation with the definitive elements, after correcting some small systematic mistakes in the first computations, left the residuals O—C for the normals as given in Table 2; the sum total of these error squares is 891, 901, 866, 874. The extreme curves, for 0·40 and 0·70, are drawn in Fig. p. 147. Applying the method of § 2 upon these results we find the mean error of k and of $\sin \vartheta_b$ for each of the values of x being given:

TABLE 2.

Primary Minimum. Residuals O—C.

Phase	Δ I	0·40 0·5471 0·2140	0·50 ·5385 ·2153	0·60 ·5298 ·2177	0·70 0·5153 ·2178	0·70 0·5174 ·2175
0·1628	+ 0·005	+ 5	+ 5	+ 5	+ 5	+ 5
1575	— ·003	— 3	— 3	— 3	— 3	— 3
1546	000	0	0	0	0	0
1508	004	— 3	— 3	— 2	— 2	— 1
1481	008	— 5	— 5	— 3	— 3	— 2
1436	017	— 8	— 9	— 7	— 7	— 6
1390	019	— 4	— 5	— 2	— 3	— 3
1345	019	+ 3	+ 2	+ 5	+ 4	+ 3
1290	030	+ 2	+ 2	+ 4	+ 3	+ 3
1252	039	0	+ 1	+ 3	0	+ 2
1204	048	+ 2	+ 4	+ 4	+ 4	+ 3
1154	062	+ 2	+ 2	+ 1	+ 3	0
1112	072	+ 4	+ 4	+ 3	+ 4	+ 3
1081	095	— 9	— 9	— 10	— 10	— 10
1050	098	— 3	— 4	— 4	— 4	— 5
1018	103	0	0	0	0	0
0978	119	— 5	— 5	— 5	— 5	— 5
0944	131	— 3	— 4	— 4	— 3	— 3
0909	139	0	— 1	— 1	— 1	0
0880	148	0	— 1	0	— 1	+ 1
0834	167	— 4	— 5	— 5	— 3	— 2
0792	167	+ 10	+ 9	+ 9	+ 10	+ 11
0762	182	+ 4	+ 4	+ 4	+ 5	+ 7
0712	210	— 6	— 6	— 6	— 5	— 3
0684	213	+ 1	+ 2	+ 1	+ 1	+ 4
0637	229	+ 1	+ 2	+ 2	+ 2	+ 5
0609	242	— 3	— 1	— 2	— 1	0
0570	249	+ 3	+ 2	+ 3	+ 3	+ 5
0548	268	— 9	— 10	— 10	— 9	— 7
0509	259	+ 11	+ 10	+ 9	+ 9	+ 11
0472	274	+ 4	+ 3	+ 2	+ 1	+ 3
0459	279	+ 2	+ 1	0	— 1	+ 1
0424	287	+ 1	0	— 3	— 4	— 2
0397	287	+ 5	+ 3	0	— 1	+ 1
0370	289	+ 7	+ 5	+ 1	0	+ 2
0329	299	0	— 2	— 5	— 5	— 3
0308	299	+ 1	0	— 3	— 4	— 2
0254	308	— 5	— 6	— 7	— 8	— 5
0200	304	+ 1	0	— 1	— 1	+ 1
0148	310	— 3	— 4	— 5	— 5	— 3
0097	302	+ 5	+ 5	+ 4	+ 4	+ 6
0044	— ·314	— 6	— 6	— 6	— 7	— 5
$\Sigma\varepsilon^2$		891	901	866	874	831
Perm.		18	23	19	25	19
Var.		15	13	15	13	17

x	k		$\sin \vartheta_b$	
x=0·40;	k=0·5471	± 0·0008;	$\sin\vartheta_b$= 0·2140	± 0·0015
0·50	0·5385	0010	0·2153	0008
0·60	0·5298	0018	0·2177	0003
0·70	0·5153	—	0·2178	0007

6. The values of $\Sigma\varepsilon^2$ found for the four cases, being nearly equal, show that x, the amount of limb darkening, cannot be determined from these data. It appears to be possible for each of these x to adjust the other elements in such a way, that they represent the observations nearly equally well. The residuals O—C in Table 2, as well as Fig. 1, show that the computed curves nearly coincide during the main part of their course; still clearer it may be seen from the computed values of Table 3.

TABLE 3.

Primary Minimum. Computed Decrease.

	x = 0·40 k = 0·5471 $\sin \vartheta_b$ = 0·2140	0·50 0·5385 0·2153	0·60 0·5298 0·2177	0·70 0·5153 0·2178	0·70 II 0·5174 0·2175
0·125	0·0404	0·0408	0·0420	0·0404	0·0405
100	1093	1089	1093	1086	1085
075	1916	1915	1917	1934	1939
050	2704	2704	2703	2708	2732
040	2922	2903	2872	2855	2876
025	3025	3016	3008	2998	3021
000	3085	3085	3085	3085	3105

The curves are divergent only at the phases near outer and inner contact, where the border parts of the disc come into play. At outer contact the differences are partly neutralized by changes in $\sin \vartheta_b$, the time of contact. Near inner contact, at $t - t_0 =$ 0·03 — 0·04^d, the curves show notable differences of figure; the differences in brightness amount to 0·006 or 0·007, corresponding to 0·009 — 0·010 magnitude. After the representation in Fig. 1 0·40 and 0·70 seem to be the extreme admissable values for x. The latter even shows such a persistence of negative O—C values, that it must be judged to be an improbable solution. Here, however, we have to consider that the basis of all these curves is the fixed value of 0·3085 as the assumed decrease in the minimum. If we take the minimum brightness 0·002 lower, then with $x = 0·70$ the representation of the lowest part of the curve can be made quite satisfactory. For this case, with a decrease 0·3105 in the minimum, we find the occulted part of the light of the larger star $\alpha_0 = 0·3314$, and the ratio k for a central eclipse 0·5174. The computation with these elements and three values of sin ϑ_b and the resulting $\Sigma\varepsilon^2$ is given in the last line of Table 1; the residuals O—C for sin $\vartheta_b = 0·2175$ are given in the last column of Table 2. Now the representation with $x = 0·70$ (called 0·70 II) is quite as satisfactory as in any other case. By an analogous but smaller change of minimum brightness $\Sigma\varepsilon^2$ in the case of $x = 0·60$ could be somewhat depressed also.

The extreme difficulty of determining the amount of limb darkening for eclipsing variables becomes manifest in these results. Even the high accuracy of the photoelectric cell is hardly sufficient; a decision between the different shapes of the light-curves near inner contact will only be possible if the number of observations in just these most sensitive parts of the curve is considerably increased. It is doubtful, therefore, whether with less accurate measures (visual estimates or photographic measures) real information about the limb darkening may be obtained. The matter is somewhat different, if photometric measures in two colors are made and compared[1]); for both series the geometric unknowns k and sin ϑ_b must be the same and cannot be adjusted separately, so that the relative limb darkening for these two colors may come out more easily.

7. The secondary minimum, theoretically, can give information about the limb darkening of the small star, through the figure of the slope of the light-curve between the constant parts. As it is improbable, regarding the small range in brightness, that a definite result about its amount of limb darkening may be reached, we have restricted ourselves to only making U- and D- solutions in this case. Since the secondary minimum affords independent data about the geometrical elements it can be used for a test of the values of k and sin ϑ_b derived from the primary minimum.

In the case of a circular orbit k (with corresponding d_0/r and i) and sin ϑ_b must be the same for both minima, and x is the only adjustable quantity. The equality of the intervals between the minima does not prove, however, that the orbit is circular, but only that the excentricity has no tangential component. There may be a radial component showing itself in the different durations of the eclipses. We can take it into account by considering sin ϑ_{b2} (ϑ = mean anomaly) as an independent unknown for the secondary minimum, and determine it by adapting the light-curve to the observations. We have then to change d_0/r at the same time. If e denotes the excentricity in radial direction and if its square is neglected, the radius vector in primary and secondary minimum is multiplied by $1 - e$ and $1 + e$, and sin ϑ in these minima is multiplied by $1 + 2e$ and $1 - 2e$. Hence in stead of the relations of § 4 we have

$$(1+k)^2 = (1-e)^2 \frac{\cos^2 i}{r^2} + (1+e)^2 \frac{\sin^2 i}{r^2} \sin^2 \vartheta_{b1} \qquad \text{(primary min.)}$$

$$\left(\frac{d_{0\cdot pr}}{r}\right)^2 = (1-e)^2 \frac{\cos^2 i}{r^2} \qquad \text{(primary min.)}$$

$$(1+k)^2 = (1+e)^2 \frac{\cos^2 i}{r^2} + (1-e)^2 \frac{\sin^2 i}{r^2} \sin^2 \vartheta_{b2} \qquad \text{(secondary min.), or}$$

$$\left(\frac{1+k}{d_{0\cdot pr}/r}\right)^2 = 1 + \left(\frac{1+e}{1-e}\right)^2 \mathrm{tg}^2 i \sin^2 \vartheta_{b1} = \left(\frac{1+e}{1-e}\right)^2 + \mathrm{tg}^2 i \sin^2 \vartheta_{b2}.$$

From these two equations $\left(\frac{1+e}{1-e}\right)^2 = 1 + 4e$ and $\mathrm{tg}^2 i$ may be computed for each k, and then for every other time during the secondary eclipse we have

$$\left(\frac{d}{d_{0\cdot pr}}\right)^2 = \left(\frac{1+e}{1-e}\right)^2 + \mathrm{tg}^2 i \sin^2 \vartheta;$$

$$\left(\frac{d}{r}\right)^2 = (1+e)^2 \frac{\cos^2 i}{r^2} + (1-e)^2 \frac{\sin^2 i}{r^2} \sin^2 \vartheta.$$

The resulting values of i and r will of course be different from the results derived from the primary minimum in the supposition of a circular orbit. The elements for which the lightcurve has been computed, and the resulting $\Sigma\varepsilon^2$ for the secondary minimum are given in Table 4; the observational data, the "reflected normals", reduced to decrease in fraction of the total light, are given in the first column of Table 5. For each k, corresponding to some x in the primary minimum, the best value of sin ϑ_{b2} was deduced by means of minimum $\Sigma\varepsilon^2$, for the uniform as well as for the darkened case. The results for the extreme cases are plotted in the lower part of Fig. p. 147. It appears from $\Sigma\varepsilon^2$ as well as from the residuals O—C in Table 5 that the representation of the data by the 10 sets of elements is not much different; the sets with $x = 0·70$ are somewhat less good than those with 0·60 or 0·50,

[1]) Cf. H. Rosenberg, *Astrophys. J.*, **83**, p. 67 (1936).

Table 4.

Secondary Minimum. Results for different hypotheses.

x_{pr} / k	$\sin\vartheta_b$ / e			$\Sigma\varepsilon^2$ U / D			$\sin\vartheta_b$ U / D	$\sin\vartheta_b\,(pr)$	e	$\Sigma\varepsilon^2$
0·40	0·240	·230	·216	530	495	685	0·232 ± ·008	0·214	0·042	490
0·5471	0·06	0·04	0·005	578	501	705	0·231 ·006	,,	0·037	500
0·50	0·240	·230	·220	532	485	548	0·231 ± ·008	0·2153	0·038	485
	0·230	·220	·210	529	558	819	0·226 ± ·008	,,	0·027	513
0·5385	0·06	0·035	0·015							
0·60	0·228	·222	·216	493	500	587	0·225 ± ·005	0·2177	0·018	486
0·5298	0·024	0·010	0·009	566	553	616	0·223 ± ·003	,,	0·017	525
0·70	0·224	·219	·214	523	518	580	0·221 ± ·003	0·2178	0·007	512
0·5153	0·014	0·003	0·004	666	605	638	0·218 ± ·004	,,	0·000	604
0·70	0·220	0·2175	0·215	537	532	565	0·218 ± ·003	0·2178	0·000	529
0·5174	0·005	0·000	0·006	618	604	614	0·217 ± ·004	,,	0·000	604

Table 5.

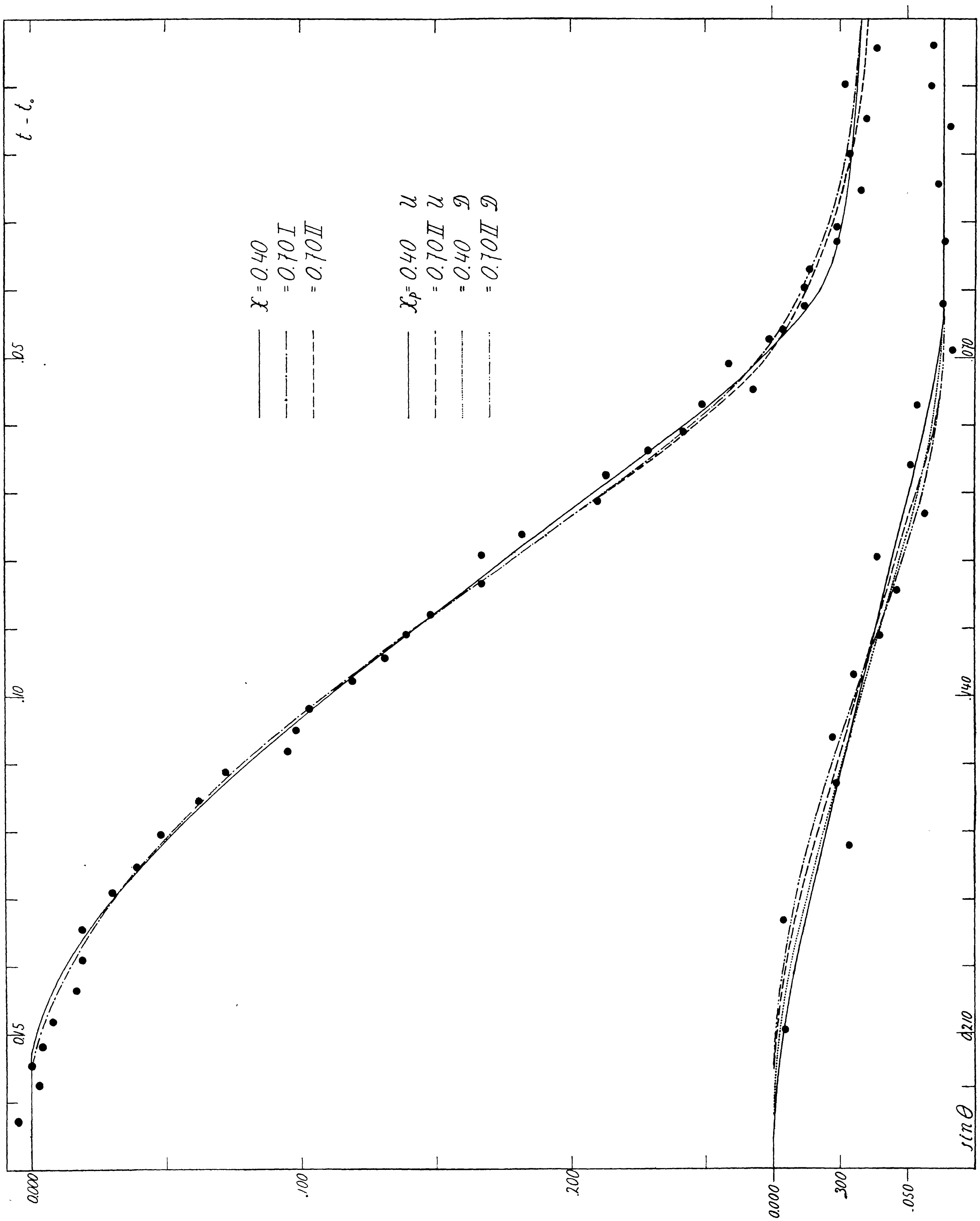
Secondary Minimum. Residuals O—C.

Phase	Δ I	0·40 U	0·40 D	0·50 U	0·50 D	0·60 U	0·60 D	0·70 I U	0·70 I D	0·70 II U	0·70 II D
0·1495	— 0·005	0	— 2	— 1	— 3	— 2	— 3	— 2	— 3	— 4	— 4
1332	004	+ 8	+ 6	+ 8	+ 4	+ 6	+ 4	+ 5	+ 3	+ 4	+ 2
1222	028	— 11	— 12	— 10	— 14	— 12	— 14	— 12	— 15	— 13	— 15
1130	024	0	— 1	0	— 2	— 1	— 3	— 1	— 3	— 2	— 3
1062	022	+ 5	+ 5	+ 6	+ 5	+ 4	+ 4	+ 5	+ 4	+ 4	+ 3
0970	030	+ 3	+ 4	+ 3	+ 4	+ 3	+ 3	+ 4	+ 4	+ 2	+ 3
0912	040	— 3	— 2	— 3	— 1	— 3	— 1	— 2	— 1	— 3	— 2
0845	046	— 5	— 3	— 4	— 2	— 4	— 2	— 3	— 2	— 4	— 2
0795	039	+ 5	+ 8	+ 6	+ 9	+ 6	+ 8	+ 8	+ 10	+ 7	+ 9
0732	056	— 9	— 5	— 8	— 5	— 8	— 5	— 5	— 3	— 7	— 4
0660	051	+ 1	+ 5	+ 2	+ 4	+ 3	+ 6	+ 5	+ 7	+ 4	+ 7
0572	054	+ 4	+ 7	+ 5	+ 6	+ 5	+ 7	+ 8	+ 8	+ 7	+ 8
0490	067	— 5	— 4	— 5	— 4	— 4	— 4	— 3	— 3	— 3	— 3
0412	063	0	0	0	0	0	0	0	0	0	0
0330	064	— 1	— 1								
0245	062	+ 2	+ 2								
0160	071	— 7	— 7	The same		The same		The same		The same	
0100	059	+ 5	+ 5								
0040	060	+ 4	+ 4								

and the D-solutions are less good than the U, indicating that the limb darkening of the small star is not large. So the secondary minimum cannot give a clear decision which of the different values of limb darkening for the large star should be preferred.

Addendum. After our computations were finished an article by Dr. A. Hnatek appeared[1]), where the same method is used, viz. to find the most probable value of an element by computing the sum total of error squares for different hypotheses. Dr. Hnatek's conclusions, however, are opposite to ours; he gives a result for the limb darkening (in the case of KR Cygni) in 3 decimals (making use of 22 normals with a mean error of 0·03^{m}), whereas we found (from 61 normals with a m.e. of 0·004^{m}) that even the first decimal was unreliable. The source of this difference may be traced firstly to his exclusion of 9 normals simply because they are deviating from the light-curve more than 0·029^{m} and retaining only the normals situated close to the curve—a procedure contrary to the principles of error theory. Further on he treats the points taken from a mean curve as if they were observed quantities; the well marked and sharp minimum of $\Sigma\varepsilon^2$ in this case can have no real significance. A second source of discrepancy is this that Dr. Hnatek takes a fixed value for the duration of the eclipse, by increasing the uniform value 0·226^{d} to the estimated amount of 0·250^{d} for all his cases. If this duration is taken constant then there is of course one solution with corresponding limb darkening that fits better than other solutions. Other assumptions as to the duration of the eclipse would have procured other values of limb darkening with nearly the same representation of observational data.

[1]) A. Hnatek, *Ueber die Bestimmung der Randverdunkelung bei Bedeckungsveränderlichen, A.N.* 6260, **261**, 361.


t - t0
.05
.10
.15
χ = 0.40
= 0.70 I
= 0.70 II
χp = 0.40 U
= 0.70 II U
= 0.40 D
= 0.70 II D
.070
.140
.210
sin θ
0.000
.100
.200
0.000
.300
.050