Astronomy and Geophysics. — *"Ionisation Equilibrium in Stellar Atmospheres and in the Earth's Atmosphere"*. By Prof. A. PANNEKOEK.

In 1920 the Bengalese physicist MEGH NAD SAHA derived from the equations of chemical reactions in thermodynamical equilibrium a formula for the ionization as a function of temperature and pressure:

$$\frac{x}{1-x} P_e = \frac{(2\pi\, m)^{3/2}}{h^3} (kT)^{5/2} e^{-\chi/kT}$$

where x is the fraction of the gas ionized, χ the ionization energy per atom, m the mass of an electron and P_e the electron pressure. This formula has already had an extensive application on the stellar atmospheres and it has provided a physical basis for the explanation of stellar spectra. There remained, however, an uncertainty. The stellar atmospheres are not gases in thermodynamical equilibrium, because a radial stream of energy goes outward causing the temperature to decrease to a boundary temperature. Thus the formula is not strictly applicable to these atmospheres; and where discrepancies are found between observational data and theory it cannot be decided whether this discrepancy is real or should be ascribed to this deficiency of theory.

This uncertainty can now be removed since formulas relating to ionization in non-isothermal gases have been derived, at first by SAHA and SUR, and then by J. WOLTJER [1]), who made use of MILNE's researches. For the unknown mechanisms of ionization by incident radiation and of recombination of an electron with an ion MILNE introduces probability coefficients, which multiplied by the numbers of the particles give the numbers of the processes of ionisation and recombination per second. In a state of equilibrium these numbers must be equal; thus a relation between the numbers of the three kinds of particles is found. Applying this relation to the case of an isothermal gas, we must find the SAHA formula; this condition determines relations between the probability coefficients that have been deduced by MILNE [2]). Now we must assume that these coefficients express atomic properties not depending on statistical distribution quantities such as temperature and pressure. Then they must hold also in other cases, when there is no equilibrium and we may introduce them into the formulas representing these cases.

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[1]) Physica **5**, p. 406.
[2]) Philosophical Magazine, (6) **47**, p. 209, 1924.

If we put x_1 x_2 y the numbers of atoms, ions, and electrons per unit volume, $I(\nu)$ the intensity of the total radiation of frequency ν falling from all sides on the atoms, the number of ionizations by this radiation may be expressed by $x_1 \psi(\nu) I(\nu) d\nu$. This probability coefficient ψ has only values for $\nu > \nu_0$, if $h\nu_0$ denotes the energy of ionization; if the energy $h\nu$ is absorbed from the radiation, the part $h\nu_0$ is used for the ionization and the rest $h(\nu-\nu_0)$ is transformed into translation energy of the electron, amounting to $\frac{1}{2} mv^2 = h(\nu-\nu_0)$. Thus a continuous absorption band is formed, from ν_0 extending to $\nu = \infty$. The compensating process is the recombination of an electron with an ion by impact; the liberated ionization energy $h\nu_0$ with the kinetic energy of the electron is radiated as a quantum $h\nu = h\nu_0 + \frac{1}{2} mv^2$; thus a continuous emission band appears. The number of recombinations, proportional to the number of ions (assumed at rest) and the number of electrons, which by the partition law of MAXWELL have this velocity v, is given by

$$8\pi x_2 y \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} \{F(v) + I(\nu) G(v)\} v^3 dv,$$

where $F(v)$ and $G(v)$ denote the probabilities of spontaneous and of induced recombinations. If the total numbers of ionizations and recombinations, caused by radiation of all wavelengths, are equal, we have equilibrium. Then

$$x_1 \int_{\nu_0}^{\infty} \psi(\nu) I d\nu = x_2 y \int_{\nu_0}^{\infty} 8\pi^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-h(\nu-\nu_0)/kT} \{F(v) + I.G(v)\} \frac{v^2 h}{m} d\nu.$$

In the case of isotropic radiation of temperature T, where I is given by PLANCK's formula, this equation must produce the SAHA equation

$$\left(\frac{x_2 y}{x_1}\right)_0 = \frac{(2\pi m)^{3/2}}{h^3} (kT)^{3/2} e^{-h\nu_0/kT}.$$

In this case the integral signs may be dropped, because for each separate frequency there must be equilibrium between ionization and recombination. In this way relations between $F(v)$ and $G(v)$, and $\psi(\nu)$ are found, which introduced into the equilibrium formula give it the form

$$\frac{x_2 y}{x_1} = \left(\frac{x_2 y}{x_1}\right)_0 \frac{\int \psi(\nu) I d\nu}{\int \psi(\nu) e^{-h\nu/kT}\left(\frac{8\pi h\nu^3}{c^2} + I\right) d\nu}$$

where the integrations are taken from ν_0 to ∞. This is the same formula as deduced by WOLTJER.

2. For the integration the function $\psi(\nu)$ is needed. It is not quite

unknown; from the experimental fact that the coefficient of absorption at the limit of the band, i.e. for $v = v_0$ or $v = 0$ is neither zero nor infinite, MILNE deduces that $Lim\ F(v)/v^2$ for $v = 0$ must have a finite value. The simplified mechanism of electron capture used by EDDINGTON to compute the stellar absorption coefficient causes a strict proportionality of $F(v)$ with v^2. This corresponds to $\psi(v)$ being proportional to v^{-3}. If this proportionality does not strictly hold, the differences become important only for large v, where I is very small. Thus the result computed with this formula, if not rigidly, will certainly be practically right.

For a stellar atmosphere we compute I from the conditions of radiative equilibrium, for which the SCHWARZSCHILD approximation suffices. This means that I for the directions from the lower half of the sphere is taken equal to black radiation of the temperature T_1, the effective temperature of the sun or the star, and for all directions from above it is taken 0. In equilibrium with this radiation the atmosphere has a temperature $T = T_1 \sqrt[4]{1/2}$. In WOLTJER's formula the second therm I in the denominator, originating from $G(v)$, may be neglected for the same reason and in the same cases as the second denominator term in PLANCK's formula; it is only relevant for low frequencies, while here we have to integrate over high frequencies only. The integrals in the above formula now become:

$$\int \psi(v)\,I\,dv = \int C v^{-3} \frac{4\pi h v^3}{c^2} e^{-hv/kT_1}\,dv = \frac{4\pi\,Ck\,T_1}{c^2} e^{-hv_0/kT_1}$$

$$\int \psi(v)\,e^{-hv/kT} \frac{8\pi h v^3}{c^2}\,dv = \int C v^{-3} \frac{8\pi h v^3}{c^2} e^{-hv/kT}\,dv = \frac{8\pi\,Ck\,T}{c^2} e^{-hv_0/kT}$$

Thus the factor by which the formula of SAHA has to be multiplied, will be

$$\tfrac{1}{2} \frac{T_1}{T} e^{-hv_0\left(1/kT_1 - 1/kT\right)} .$$

If we write χ for the ionization energy hv_0, the formula becomes

$$\frac{x}{1-x} P_e = \tfrac{1}{2} \frac{(2\pi\,m)^{3/2}\,k^{5/2}}{h^3}\,T_1\,T^{3/2}\,e^{-\chi/kT_1}.$$

Thus we find that the ionization, if computed after the Saha formula not with the atmospheric but with the effective temperature, needs only a constant correction factor $\tfrac{1}{2}(T/T_1)^{3/2}$, which in this case is $2^{-11/8}$ [1]).

3. In the same way also the ionisation of the upper layers of the atmosphere of the earth may be computed. From the propagation of wireless waves around the spherical earth it has been deduced that the highest layers of our atmosphere are conductive through the presence of a great number of free electrons or ions. The lower limit of this HEAVISIDE

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[1]) The ionization by electron impacts for such small pressures, after R. H. FOWLER's formula (Philos. Magazine, (6) **47**, p. 275), is wholly negligible compared with the photoelectric ionization.

layer is given usually at 90 kms and its electron density at 10^5 per ccm. at least; in the daytime the limit seems to be lower and the whole phenomenon seems to be less regular than by night. In order to explain this layer either ionization by impacts of swift moving particles is assumed, which ejected by the sun penetrate our atmosphere, or photoelectric ionization by the solar radiation. The latter effect can now be computed exactly.

The total radiation falling upon the particles is solar radiation proceeding from $^1/_{184000}$ of the sphere; this fraction will be called β. We assume a single gas with ionisation energy χ and temperature T; the effective temperature of the radiation is T_1. Then we have in the same way as in the former case

$$K = \frac{x^2}{1-x^2}\, p = \beta \cdot \frac{(2\pi m)^{3/2}\, k^{5/2}}{h^3}\, T_1\, T^{3/2}\, e^{-\chi / kT_1}\,,$$

where in the first part the total pressure has been introduced, because the numbers of electrons and ions are equal. The gases to be dealt with are nitrogen, oxygen, and hydrogen, for which the ionization potential for the molecules has to be taken. For nitrogen it is 16.9 after the measures of BRANDT; for hydrogen 16.1 ± 0.2 is given by FRANCK and JORDAN[1]) as an average of determinations by many observers; for oxygen, where 15.5 resulted from the older measures of MOHLER and FOOTE a recent accurate determination by LOCKROW[2]) also gives 16.1. Assuming a temperature $-55°$ ($T = 218°$ K) we find for $\log K$ values -18.65 for N, -17.98 for H and O. Thus in all practical cases x is very small, $1 - x^2$ may be taken 1, and for K we may write $x^2 p$.

The energy used for ionizing the molecules is taken from the solar radiation having a frequency higher than $\nu_0 = eV/300\,h$; thus an absorption band is formed, beginning at $\lambda = 730\,A$ (for N) or $= 766\,A$ (for O and H). The absorption caused by the ionization of the higher layers weakens the radiation falling upon the lower layers: therefore the ionization of these lower layers is diminished also, and disappears below a certain limit, when all active wavelengths have been extinguished from the solar radiation. Putting α for the atomic absorption coefficient, N' for the number of atoms per ccm for one atmosphere and a temperature T, we find that a layer of the gas 1 cm thick absorbs the fraction $\alpha N' p$ of the incident light. For the atomic absorption coefficient α a formula has been deduced by MILNE by making use of KRAMERS' computations; by omitting several factors of order zero it becomes

$$\alpha\,(\nu) = \frac{16\pi^2}{3\sqrt{3}}\, \frac{Z^2 e^6}{c\,h}\, \frac{\chi}{(h\nu^3)}.$$

It has been derived for atoms; since factors of order zero are irrelevant here it may be used also for molecules. Computing its value for the limit

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[1]) Anregung von Quantensprüngen durch Stösse (Handwörterbuch der Physik **23**, 749, 757).

[2]) Astrophysical Journal **63**, p. 205 (1926).

of the absorption band $\nu = \nu_0$ we find $log\ a = -15.91$ for N, -15.71 for O, -17.56 for H.

The distribution of the ions over different layers of the atmosphere may now be computed. In a layer dh we have $N'p\,dh$ atoms; thus the absorption is $dI/I = aN'p\,dh = -aN'H\,dp$, if with H we denote the aequivalent height of one atmosphere. Then

$$I/I_0 = e^{-\alpha N'Hp}.$$

gives the weakening of the active solar radiation. This factor adds to the coefficient β in the ionization formula; the ionization is given by

$$x^2 p = K e^{-\alpha N'Hp}$$

and the number of electrons per ccm $n = x N'p$ becomes

$$n = N' \sqrt{pK}\, e^{-1/2\alpha N'Hp}.$$

Introducing the numerical values $log\ N'$ (for $T = 218°) = 19,83$; $log\ H = 5,82$ for nitrogen, 5,76 for oxygen, 6,96 for hydrogen. and $log\ K$ as given above, we have

$$n = [10.51] \sqrt{p}\, e^{-[9.44]p} \quad \text{(for nitrogen)}$$

$$n - [10.84] \sqrt{p}\, e^{-[9.58]p} \quad \text{(for oxygen)}$$

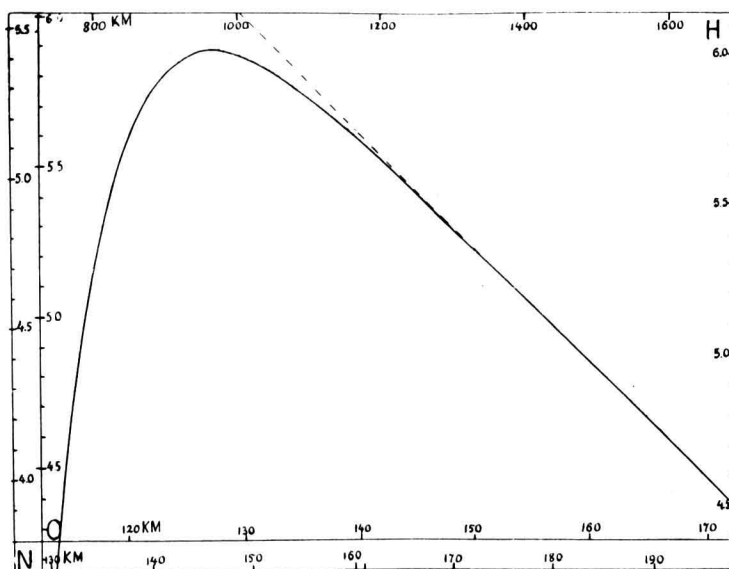$$n = [10.84] \sqrt{p}\, e^{-[8.93]p} \quad \text{(for hydrogen). [1]}$$

In the following table the value of $log\ n$ has been computed for different values of the pressure p, and the corresponding heights in kilometres have been added.

Number of electrons per ccm.

$log\ p$ in atm.	Nitrogen		Oxygen		Hydrogen	
	$log\ n$	h	$log\ n$	h	$log\ n$	h
− 7					− 30	500
− 8	− 5.5	120	− 10	103	+ 3.14	710
− 9	+ 4.81	135	+ 4.68	116	+ 5.97	920
− 10	+ 5.39	150	+ 5.67	129	+ 5.80	1130
− 11	+ 5.00	165	+ 5.32	142	+ 5.34	1340
− 12	+ 4.51	180	+ 4.84	155	+ 4.84	1550
− 13	+ 4.01	195	+ 4.34	168	+ 4.34	1760
− 14	+ 3.51	210	+ 3.84	182		

[1] These computations are made for the limit of the band ν_0. For greater values of ν the absorption is smaller, thus the intensity of the smaller wave lengths increases relative to this limit in the lower atmospheric layers. Computation shows, however, that this difference becomes perceptible only in depths where all action has disappeared.

The figure shows the variation of this number *log n* with the height, which for each gas is represented by the same curve, only referred to

different scales, which for each of them has been indicated [1]). The dotted line represents the number of electrons if there had been no absorption. The figure shows that, descending from great heights the number of electrons at first increases regularly, then reaches a maximum between 10^5 and 10^6, and falls rapidly to zero because of the increasing absorption. For nitrogen and oxygen the maximum occurs *at* 145 or 128 kms, the lower limit at 130 and 113 km.

Thus the number of electrons corresponds very well with what has been deduced from wireless experiments. The lower limit, however, found here, lies at a greater height than is usually adopted. Now in the newest literature greater heights are sometimes given; from experiments with short waves TUVE and BREIT deduced heights between 80 and 160 kms. From the experiments of TAYLOR and HULBURT and others HESS concludes: "Die Elektronenzahl pro Kubikzentimeter scheint nach diesen Untersuchungen in Höhen von 110 bis 250 km. ihren Maximalwert (10^5 pro ccm.) zu erreichen (bei Tage)" [2]). According to our computations electrons by photoelectric ionizations cannot occur below 100 kms. A moderate error in the adopted atomic coefficient of absorption has no appreciable influence on log *n* as a function of *p*; the value a should be decreased 100 times to lower the limit 30 kms. A lower temperature than —55° C. for the upper atmospheric, however, which contracts the

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[1]) The scale for O should be raised by 0,20; the values used in constructing the figure had been computed with an ionization potential of 15,5.

[2]) V. T. HESS, Die elektrische Leitfähigkeit der Atmosphäre, p. 164.

gases more at the bottom of the atmosphere, will diminish the height corresponding to each p appreciably. If the temperature decreases regularly from —55° C. at 15 kms to —160° C. at 90 kms, the lower limit of the electron layer would be brought 30 kms lower. A considerable amount of electrons at a height of 90 kms or lower, in as much as it cannot be ascribed to ionization by impacts, thus points to a much lower temperature of these upper layers than is usually assumed.

For hydrogen the region of ionization begins only above 700 kilometers. To the HEAVISIDE layer ionization of hydrogen cannot contribute anything. Conversely the ionization of hydrogen at great heights extinguishes all solar wave lengths below 766 A., just the frequencies that are necessary for the ionization of nitrogen and oxygen. The existence of a hydrogen atmosphere above the nitrogen and oxygen atmosphere makes the existence of a HEAVISIDE layer by photoelectric ionization impossible. Therefore, unless this HEAVISIDE layer may be explained wholly by other causes, such as impacts by particles of high velocity, its existence shows, that hydrogen in the usually adopted quantity cannot occur in our atmosphere.

Postscript. From the refraction of sound waves in the higher layers of the atmosphere Prof. E. VAN EVERDINGEN in 1915[1]) concluded that the amount of hydrogen in our atmosphere usually adopted is too high; a partial pressure of 10^{-6} at the surface fits better to the observed outer radius of the silent region. This is still too much to allow ionization of nitrogen and oxygen. With a somewhat larger percentage of helium and absence of hydrogen it seems possible, however, to represent the data on the propagation of sound in the atmosphere.

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[1]) These Proceedings Vol. **18**, p. 933.