

The Theoretical Contours of Absorption Lines.

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1. The problem of determining the contours of absorption lines in the solar spectrum and in stellar spectra has been treated in different papers by Schwarzschild, Milne, Eddington, Unsöld, etc. Schwarzschild and Schuster derived formulæ for the intensity of the radiation emitted from the outer layers of an atmosphere in the case of black absorption and emission as well as in the case of diffusion (scattering). From the visibility of the Fraunhofer lines at the border of the sun Schwarzschild concluded that diffusion plays an important part in the production of these lines. Unsöld * applied his formulæ in the case of pure diffusion to the wings of the broad lines of Ca^+ and H , by making use of Voigt's formulæ for the absorption coefficient; this treatment gives zero for the central intensity, at variance with observation. Milne † showed that the diffusion formulæ are valid in the case of monochromatic equilibrium, whereas the collisions of electrons with the radiating atoms introduce black-body emission and absorption into the formulæ. Because an exact solution presents mathematical difficulties he gives a solution for the case of a constant ratio of the diffusion and the absorption coefficients. This solution forms the basis of his computations concerning ionization effects on line intensities. Eddington's formulæ ‡ are based on the same supposition, but he takes account of the oblique directions in which the radiation traverses the atmosphere. Unsöld § has made use of Milne's solution in deriving results on central intensities; Woolley || has done the same, using Eddington's formulæ.

In these results the residual intensity depends only on the ratio of the diffusion and the black absorption coefficients; their absolute values do not enter into the formulæ. This ratio is assumed to be constant for the whole atmosphere; so the result is obtained that in the centre of a Fraunhofer line, where the general absorption may be neglected, the intensity is the same for the members of a multiplet (Unsöld), and depends on a certain average pressure (Woolley). In reality this ratio is not constant because the absorption coefficients increase with pressure, *i.e.* with the depth in the atmosphere; hence the residual intensity must be smaller for larger absolute values of the coefficients, because the depth from which the radiation reaches the surface decreases with increasing absorption.

Eddington expressed strong doubts whether, through the complications of the problem in its physical as well as its mathematical and astronomical foundations, the results obtained by these approximations so far have any real value. To remove such doubts it will be necessary

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to work out exactly the consequences of strictly defined suppositions ; only when the lack of harmony with observation cannot be ascribed to the approximate mode of treatment, shall we be able to trace its origin back to the physical and astronomical foundations. Therefore an attempt is made in this paper to find an exact solution of the equations representing existing conditions, though simplified, as nearly as possible.

2. The equations are to represent the following conditions and suppositions :—

(a) In the atmosphere only two vertical streams of energy are assumed to exist, one going outward, one inward. Schwarzschild, for the case of diffusion, Milne, for the case of absorption, have derived exact formulæ for radiation passing through the atmosphere in every direction. The results deviate only slightly from the simplified case, if only certain numerical factors ($1/2$, $3/2$) are added to the absorption and diffusion coefficients and the optical depth, to take account of the average inclination of the energy streams. A comparison of Eddington's result for the residual intensity with Milne's, if expressed by the same symbols, shows that here the influence of oblique radiation only introduces a coefficient $2/3\sqrt{3} = 1.15$ into some of the terms. Therefore these factors will be omitted in our formulæ.

(b) We consider one kind of atoms having two energy levels, for which the difference of energy corresponds to the frequency ν_0 ; we consider only frequencies ν slightly different from ν_0 , belonging to the realm of the absorption line or to the part of the continuous background adjacent to it. The exchanges of energy to be considered here are : 1a, the change of radiation energy of the field into quantum energy through absorption by the atom ; 1b, the reverse, change of quantum energy into radiation through emission by the atom ; 2a, change of translation energy of electrons into quantum energy by inelastic collisions ; 2b, change of quantum energy of the atom into translation energy by hyperelastic collisions. Milne * derived the equation for the radiation of frequency ν ; putting n_1 and n_2 the number of atoms in state 1 and 2 per unit volume, $B_{12}I(\nu)$ the probability of transition 1 \rightarrow 2 by incident radiation $I(\nu)$, b_{12} the probability of the same transition by electron collisions (for which a Maxwellian velocity distribution with temperature T is assumed), $E(\nu, T)$ the black-body radiation according to Planck's formula, $\eta = b_{12}/B_{12}E(\nu, T)$ the ratio of the number of transitions by electron collisions and by absorption of black radiation of the same temperature T , we have

$$\frac{dI(\nu)}{(n_1 - n_2)B_{12}h\nu dx} = -I + \frac{\int I(\nu) \frac{d\omega}{4\pi} + \eta E(\nu, T)}{1 + \eta}.$$

Substituting macroscopic quantities for the atomic values

$$(n_1 - n_2)B_{12}h\nu = s_0\rho = (k + s)\rho ; \quad \eta = k/s,$$

* "The Effect of Collisions on Monochromatic Radiative Equilibrium," *M.N.*, **88**, 493 (1928).

this equation becomes

$$\frac{dI(\nu)}{\rho dx} = - (k + s)I(\nu) + s \int I(\nu) \frac{d\omega}{4\pi} + kE(\nu, T).$$

Here it appears that the complete monochromatic absorption coefficient s_0 is divided into two parts ; the part s is emitted directly by the atom and works as a coefficient of diffusion ; the part k is transformed into energy of electron translation (acting as black matter) and is returned as black radiation.

(c) There is a general absorption and emission for all wave-lengths, which produces the continuous background of the spectrum. We suppose it to be produced by direct transformation of radiation into translation energy, and *vice versa* at electron collisions with atoms of all kinds : $\frac{1}{2}m(v_1^2 - v_2^2) = h\nu$. The coefficient of this general absorption k_0 is assumed to be proportional to $PT^{-3/2}$ (P = electron pressure), according to Gaunt's formula. In the centre of an absorption line it is negligible compared with the line absorption s_0 ; in the wings of course it is important.

(d) The probability coefficient of collisions b_{12} is in the same way assumed to be proportional to the electron pressure ; thus we have $\eta = k/s \sim P$.

(e) The total absorption in each space element of the atmosphere, which determines the temperature distribution, is composed of this general absorption and the sum total of all line absorptions. After Minnaert the latter may be estimated as 15 per cent. of the total absorption. Introducing a homogeneous depth $d\xi = -\rho dh$, and an optical depth τ by $d\tau = \bar{k}d\xi$, where \bar{k} is the total absorption coefficient $k_0 + k$ for all ν , we have for the total black-body radiation $E = E_0(1 + \tau)$.

(f) We assume a constant composition of the atmosphere. Differences of ionization at different depths will not be considered in this paper.

3. We first treat the continuous spectrum background ; for these wave-lengths outside the lines we suppose that only the general absorption with coefficient k_0 is working. For the intensity I and I' of the two vertical streams of energy we have

$$\frac{dI}{d\xi} = k_0I - k_0E ; \quad \frac{dI'}{d\xi} = -k_0I' + k_0E.$$

Introducing $d\tau = \bar{k}d\xi$, and putting $k_0/\bar{k} = n$, not varying with the depth, and further putting $I + I' = y$, $I - I' = z$, we have

$$\frac{dy}{d\tau} = nz, \quad \frac{dz}{d\tau} = n(y - 2E).$$

By means of the boundary conditions $y_0 = z_0$ for $\tau = 0$, and $y = 2E$ for $\tau = \infty$, we find the radiation leaving at the surface

$$I_0 = y_0 = z_0 = \int_0^\infty E(\tau)e^{-n\tau}n d\tau.$$

This holds for each wave-length λ , frequency ν , outside the absorption lines. The emission function $E(\lambda, \tau)$ depends on τ through the temperature T , for which $T^4 = T_0^4(1 + \tau)$. To a first approximation we may put $E(\lambda, \tau) = E(\lambda, 0)(1 + c\tau)$, where c is written for $\frac{1}{4}h\nu/kT = \frac{1}{4}c_2/\lambda T$. For $T = 6000^\circ$ this coefficient is 1 for $\lambda = 5950 \text{ \AA}$.; for $\lambda 4000$ it is nearly 1.5. Then the integration gives

$$I_0 = E(\lambda, 0) \left(1 + \frac{c}{n}\right) = E(\lambda, 0)(1 + ck/k_0). \quad (1)$$

If we had taken $c = 1$ and neglected the difference between the general and the total absorption the net outward stream of energy of wave-length λ would have been $F(\lambda) = 2E(\lambda, 0)$. The deviation in our formula expresses first the fact, that for shorter wave-lengths, where $c > 1$, the contrast between the continuous spectrum and the absorption lines is stronger than for longer wave-lengths; and, secondly, that this contrast is enhanced by the Fraunhofer lines contributing to the total absorption, whereby the temperature gradient is increased and emission from hotter depths reaches the surface and is radiated.

4. For a wave-length within the absorption line we have both an absorption and a diffusion term. The equations for the two vertical energy streams are now

$$\begin{aligned} \frac{dI}{d\xi} &= (k + s)I - kE - \frac{1}{2}s(I + I'), \\ \frac{dI'}{d\xi} &= -(k + s)I' + kE + \frac{1}{2}s(I + I'). \end{aligned}$$

Again introducing $y = I + I'$, $z = I - I'$ we get our fundamental formulæ

$$\frac{dy}{d\xi} = (k + s)z; \quad \frac{dz}{d\xi} = k(y - 2E). \quad (2)$$

In the general case this k consists of the general absorption k_0 and the collision part of the line absorption $(\eta/1 + \eta)s_0$. (In Eddington's paper η denotes our s_0/k_0 , his ϵ denotes our $k/s_0 = \eta/(1 + \eta)$; his dx is our $k_0 d\xi$; the other differences are the numerical coefficients produced by the oblique radiation, which would change the second members in our formulæ into $\frac{3}{2}(k + s)z$ and $2k(y - 2E)$.)

In the central parts of a Fraunhofer line the general absorption may be neglected compared with the line absorption. We will first treat this simpler problem. Now $k + s$ in (2) is equal to s_0 , and we will put $s_0 d\xi = \bar{d}x$ (special optical depth). Since the variable absorption coefficients depend on electron pressure we have to express the general optical depth τ that determines the temperature gradient, in this special co-ordinate x , by means of their relations to the electron pressure.

In $dx = s_0 d\xi = -s_0 \rho dh$, ρ denotes the total density, and the relative abundance of the atoms producing our Fraunhofer line is included in the line-absorption coefficient s_0 . To introduce the electron pressure P we substitute $\rho dh = -dp/g$, and for the ratio of total

pressure to electron pressure, which we suppose to be constant through the atmosphere (*vide* 2 (f)), we write $p/P = dp/dP = p'$. Then we have $d\xi = p'dP/g$. The general absorption coefficient k_0 is assumed to be proportional to P , thus $k_0 = \kappa_0 P$. The same holds then for k , which chiefly consists of k_0 . Putting $k = \bar{k}P$, we have

$$d\tau = k d\xi = \bar{k} \frac{p'}{g} P dP; \quad \tau = \frac{\bar{k}}{2} \frac{p'}{g} P^2.$$

Introducing now the special optical depth x , we have

$$dx = s_0 d\xi = s_0 \frac{p'}{g} dP; \quad P = \frac{g}{p' s_0} x; \quad \tau = \frac{\bar{k}}{s_0^2} \frac{g}{p'} \frac{x^2}{2}.$$

We treat κ_0 and \bar{k} as constants; this means that in going downward in the atmosphere the change of the absorption coefficient with the increasing electron pressure is represented in our formulæ, but not its variation caused by the increase of temperature at greater depths.

The black-body emission E can now be expressed in terms of x ,

$$2E = 2E(\lambda, 0)(1 + c\tau) = 2E(\lambda, 0) \left(1 + \frac{1}{2} c \frac{\bar{k}}{s_0^2} \frac{g}{p'} x^2 \right).$$

Putting

$$\frac{\kappa_0}{s_0^2} \frac{g}{p'} = \alpha \quad . \quad . \quad . \quad . \quad (3)$$

this is transformed into

$$2E = 2E_0 \left(1 + \frac{1}{2} c \frac{\bar{k}}{\kappa_0} \alpha x^2 \right).$$

For the division of the line absorption into a monochromatic and a black-body part we introduce two quantities Q and q by

$$\frac{k}{s} = \frac{P}{Q} = \frac{x}{q}; \quad q = \frac{p'}{g} s_0 Q = s_0 \frac{Q}{P} \int \rho dh \quad . \quad . \quad (4)$$

Q is the electron pressure, q the corresponding special optical depth for which half the energy of the excited atom is removed by collisions, half is radiated, *i.e.* for which $\eta = 1$. Q may be considered as a constant for a certain line; q is proportional to s_0 and varies rapidly over the absorption line, being very large in the centre and decreasing to 0 at the extreme wings; neither of them changes with the depth of the layer. Introducing these quantities the equations (2) take the form

$$\frac{dy}{dx} = z; \quad \frac{dz}{dx} = \frac{x}{x+q} \left\{ y - 2E_0 \left(1 + \frac{1}{2} c \frac{\bar{k}}{\kappa_0} \alpha x^2 \right) \right\} \quad . \quad (5)$$

For the wings of a Fraunhofer line the only difference is that in k we have to add the general absorption $k_0 = \kappa_0 P = \frac{\kappa_0}{s_0} \frac{g}{p'} x$. Then the equations

$$\frac{dy}{d\xi} = (k_0 + s_0)z; \quad \frac{dz}{d\xi} = \left(k_0 + s_0 \frac{x}{x+q} \right) (y - 2E);$$

take the form

$$\frac{dy}{dx} = (1 + \alpha x)z; \quad \frac{dz}{dx} = \left(\frac{x}{x+q} + \alpha x \right) \left\{ y - 2E_0 \left(1 + \frac{1}{2} c \frac{\bar{\kappa}}{\kappa_0} \alpha x^2 \right) \right\} \quad (6)$$

An alternative form of these equations is obtained by putting $\alpha x = u$:

$$\frac{dy}{du} = \frac{1}{\alpha}(u+1)z; \quad \frac{dz}{du} = \frac{1}{\alpha} \left(u + \frac{u}{u+\alpha q} \right) \left\{ y - 2E_0 \left(1 + \frac{1}{2} c \frac{\bar{\kappa}}{\kappa_0} \frac{u^2}{\alpha} \right) \right\} \quad (6^*)$$

The boundary conditions, for (5) as well as for (6), are that at the surface, for $x = 0$, $y = z$, and at great depths, for $x \rightarrow \infty$, $y - 2E$ disappears.

5. The equations (5) and (6) cannot be integrated in a finite form. For small values of x development in series is possible; since, however, one of the boundary conditions stands for $x = \infty$, this cannot give a practical solution. After trying different means, the least laborious method was found to consist in simple mechanical integration, beginning with assumed values of $y_0 = z_0$ for $x = 0$, and varying them till the desired result for $x \rightarrow \infty$ was obtained. With appropriate small steps, Δx , in x the equations (5) take the form

$$y_{n+1} = y_n + z_{n+1/2} \Delta x; \quad z_{n+1/2} = z_{n-1/2} + \frac{x_n}{x_n + q} \Delta x (y_n - 2E)$$

and y_{n-1} , $z_{n-1/2}$, y_n , $z_{n+1/2}$, y_{n+1} , etc., are computed in succession. The variables y , z and their common zero value $y_0 = z_0$ consist of two terms, one expressed in $2E_0$, the other in $E_0 c \frac{\bar{\kappa}}{\kappa_0} \alpha$; the numerical coefficients are calculated separately. The first-named one approaches, with increasing x , to 1 for y , to 0 for z ; if the initial zero value y_0 is too large, then the numerical y reaches 1 and passes this value, while z returns and begins to increase before reaching 0; if y_0 is too small, then x reaches 0 and passes it, while y returns and decreases before reaching 1. At great depths the values belonging to adjacent y_0 are strongly diverging; thus by repeated interpolation it is possible to find the right y_0 with sufficient precision. In the second term the numerical value of $y - x^2$ has also to decrease asymptotically to 0; if y_0 deviates from the true value, $y - x^2$ either returns and goes up increasingly before reaching 0, or passes through 0, decreasing with growing rapidity.

The numerical results of the solution of equations (5) are given in Table I. It is extended to low values of q , though here the general absorption comes into play and equations (6) should be used. The second term is expressed in $E_0 c \frac{\bar{\kappa}}{\kappa_0} \alpha$, where α , because it contains $1/s_0^2$,

is proportional to the inverse square of q . The product $E_0 c \frac{\bar{\kappa}}{\kappa_0} \alpha q^2$ is independent of s_0 and constant over the breadth of the absorption line; hence the second term is given in Table I. also as the numerical coefficient of this constant. Here it appears that for large values of q , *i.e.* for strong absorptions in the central parts of the line, the influence

of the second term disappears, because only radiation from the uppermost layers of constant temperature is emitted from the surface.

TABLE I.

q .	Coeff. of E_0 .	Coeff. of $E_0 e^{\frac{\bar{\kappa}}{\kappa_0} \alpha}$.	Coeff. of $E_0 e^{\frac{\bar{\kappa}}{\kappa_0} \alpha q^2}$.
0	1.0	1.0	..
0.1	.877	1.141	1.14.1
0.5	.808	1.383	5.53
1	.744	1.67	1.67
5	.551	2.58	.103
10	.486	3.35	.0335
50	.325	5.98	.00239

6. For very large values of q direct computation becomes difficult. In this case x remains small and the second term is imperceptible. Because in this case the coefficient $x/(x+q)$ may be replaced by x/q a direct solution of equation (5) in the form of a definite integral is possible. The equation

$$\frac{d^2 y}{dx^2} = \frac{x}{q}(y - 2E_0)$$

has the solution

$$y - 2E_0 = A \int_0^1 (e^{tp} + e^{-tp})(p^2 - 1)^{-5/6} dp + B \int_1^\infty e^{-tp}(p^2 - 1)^{-5/6} dp,$$

where $t = \frac{2}{3}q^{-1/2}x^{3/2}$. The boundary condition $y - 2E_0 = 0$ for $t = \infty$ makes $A = 0$. Then we have for z ,

$$z = \frac{dy}{dx} = -Bq^{-1/2}x^{1/2} \int_1^\infty e^{-tp}(p^2 - 1)^{-5/6} p dp,$$

which by the substitution $pt = u$ becomes

$$z = -B \left(\frac{3}{2q} \right)^{1/3} \int_t^\infty e^{-u}(u^2 - t^2)^{-5/6} u du.$$

For $x = 0$, $t = 0$ these integrals are

$$y_0 - 2E_0 = B \int_1^\infty (p^2 - 1)^{-5/6} dp = BI_1$$

$$z_0 = -B \left(\frac{3}{2q} \right)^{1/3} \int_0^\infty e^{-u} u^{-2/3} du = -Bq^{-1/3} I_2.$$

The other boundary condition $y_0 = z_0$ determines B and y_0 ,

$$B = -\frac{2E_0}{I_1 + q^{-1/3} I_2}; \quad y_0 = z_0 = 2E_0 \frac{I_2}{I_1 + I_2 \sqrt[3]{q}}.$$

From the two numerical integrals I_1 and I_2 appearing in these formulæ,

one is a gamma function, and the other can be calculated easily by numerical integration,

$$I_2 = \sqrt[3]{\frac{3}{2}} \Gamma\left(\frac{1}{3}\right) = 3.066$$

$$I_1 = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt[3]{\cos \phi \sin^2 \phi}} = 4.106 = 1.339 \times 3.066.$$

Hence the limiting value of $y_0 = z_0$ for large values of q is given by

$$y_0 = z_0 = \frac{2E_0}{1 + 1.339 \sqrt[3]{q}} \quad (7)$$

Comparing this formula with some values computed by numerical integration, we find for

	Numerical.	Formula.	Difference.
$q=5$.566	.608	-.042
10	.486	.515	-.029
20	.413	.432	-.019
50	.325	.337	-.012

Thus it appears that for $q > 50$ the formula can be used without great error.

It may be remarked that for very strong absorption coefficients the residual intensity varies as the inverse third root of the coefficient of absorption. With only a constant general black absorption Milne found the residual intensity varying as the inverse square root of the line absorption.

7. The solution of formulæ (6), which have to be used for the wings of the absorption line, is little more laborious than that of formulæ (5). It takes, however, more time because now the second variable parameter α is not restricted to a second independent term, and for each different α a new series of solutions has to be made. For one absorption line we may assume the product αq^2 to be constant:

$$q = \frac{p'}{g} s_0 Q; \quad \alpha = \frac{\kappa_0}{s_0^2} \frac{g}{p'}; \quad \alpha q^2 = \kappa_0 \frac{p'}{g} Q^2;$$

the rapidly changing s_0 has disappeared. The quantity αq^2 may be written $k_0(Q) \times \xi(Q)$, *i.e.* the coefficient of general absorption at electron pressure Q (for which $\eta = 1$), multiplied by the homogeneous depth ξ for which the electron pressure is Q . These factors depend on physical probability coefficients, different for different atoms and lines, and contain, moreover, parameters of the star. Hence different sets of solutions must be made for different values of αq^2 .

In order to find the order of magnitude of this αq^2 we have made an estimate of k/s , the ratio of the number of effective collisions and the number of free transitions to the lower state for an excited atom. For the first-named number we put $\pi n_e v_e \sigma_a^2$ (number and velocity of the

electrons, effective radius of the atoms), for the other $1/\tau$ (mean lifetime). With the numerical values (for $T = 6000^\circ$) $n_e = 1.2 \times 10^{12}P$, $v_e = 4.3 \times 10^7$, $\sigma = 10^{-8}$, $\tau = 10^{-8}$, we find $k/s = 1.6 \times 10^{-4}P$; hence Q , the electron pressure, for which $k/s = 1$, is nearly 10^4 (*i.e.* 10^{-2} atmos.). The general absorption in the sun's photosphere was taken from Milne's computation *; from the value 0.78×10^4 , standing for $P = 10^2$, we derive $k_0(Q) = 0.8 \times 10^6$. The other factor $\xi(Q) = p'Q/g = p(Q)g$, the total pressure at that Q level, divided by the gravity, 3×10^4 ; since this total pressure is somewhat larger than the electron pressure Q itself, this second factor is of the order 1. Hence for the order of magnitude of the quantity aq^2 we find 10^6 in the case of solar conditions. For giant stars with smaller g the second factor will be larger; for hotter stars the first factor, in consequence of the smaller absorption coefficient, may be smaller. It should be considered, moreover, that especially owing to the uncertainty of σ , the "effective" atomic radius, our estimate could only be a very rough one.

The results of the numerical integration of (6) are found in Table II.; for $aq^2 = 10^6$ they have been computed most completely; for $aq^2 = 1$ and $aq^2 = 10^8$ only for a smaller number of values, to see the variation of the curve. The coefficients of E and of $E \frac{\bar{\kappa}}{\kappa_0} c$ are given separately, in order that the resulting intensity may be computed for different values of c (wave-length and temperature) and different suppositions for $\bar{\kappa}/\kappa_0$. That the Fraunhofer lines in total take away 15 per cent. of the spectral energy does not mean that κ_0 amounts to 85 per cent. of the total absorption coefficient $\bar{\kappa}$; for the strongest monochromatic absorptions act on radiations already strongly absorbed. Hence the quotient $\bar{\kappa}/\kappa_0$ probably is considerably larger than $100/85$, but its real amount is quite uncertain. Simply in order to obtain results, which can be used as examples for further deductions, I have computed the residual intensities r for $c\bar{\kappa}/\kappa_0 = 1.5$ (*i.e.* for the sun, with $\bar{\kappa}/\kappa_0 = 1$, $\lambda = 4000$ A.; for $\bar{\kappa}/\kappa_0 = 1.5$, $\lambda = 6000$ A.); they are expressed with the continuous background intensity $E_0(1 + c\bar{\kappa}/\kappa_0)$ taken = 1.

TABLE II.

q .	a .	Coeff. of E_0 .	Coeff. of $E_0 c \bar{\kappa}/\kappa_0$.	r .
	$aq^2 = 1$			
0.5	400	0.985	0.953	0.966
1	100	0.971	0.911	0.931
2	25	0.947	0.833	0.878
5	4	0.883	0.650	0.744
10	1	0.808	0.451	0.594
2	0.25	0.715	0.256	0.439
5	0.04	0.586	0.085	0.286
10	0.01	0.497	0.031	0.217

* "Absorption Coefficients and Pressure of Radiation," *M.N.*, 85, 773, 1925.

TABLE II.—*continued.*

q .	$aq^2 = 10^6$	a .	Coeff. of E_0 .	Coeff. of $E_0 c\bar{\kappa}/\kappa_0$.	r .
10		10^4	0.995	0.992	0.993
10^2		10^2	.958	.924	.938
2×10^2		25	.920	.858	.883
5×10^2		4	.829	.700	.752
10^3		1	.702	.527	.597
2×10^3		.25	.574	.340	.434
5×10^3		.04	.384	.150	.244
10^4		.01	.268	.0700	.149
2×10^4		25×10^{-4}	.180	.0304	.0903
5×10^4		4×10^{-4}	.103	.943 10^{-2}	.0470
10^5		10^{-4}	.0675	.376 10^{-2}	.0292
10^6		10^{-6}	.0182	.150 10^{-3}	.0074
10^7		10^{-8}	.00697	.388 10^{-5}	.0028
10^8		10^{-10}	.00315	.87 10^{-7}	.0013
	$aq^2 = 10^8$				
10^3		10^2	0.958	0.924	0.938
10^4		1	.714	.528	.602
10^5		10^{-2}	.268	.0704	.149
10^6		10^{-4}	.0657	.391 10^{-2}	.0287
10^7		10^{-6}	.0152	.180 10^{-3}	.0062
10^8		10^{-8}	.00395	.691 10^{-5}	.0016
10^9		10^{-10}	.00148	..	.00059
10^{10}		10^{-12}	.000686	..	.00027

The numerical results show that for moderate absorptions the residual intensity depends only on a , and the differences of q have hardly any influence. Even for the widely different supposition of $aq^2 = 1$ the two separate coefficients deviate, but the resulting intensity does not. This means that the collision part of the monochromatic coefficient in these cases is negligible compared with the general absorption. At the bottom of the tables, however, for very low residual intensities, the matter is different; here the values for the same a decrease with increasing aq^2 and they approach to a dependence on q chiefly. The low residual intensities themselves depend only on the uppermost atmospheric layer; the value of r , however, depends on $c\bar{\kappa}/\kappa_0$ because it is the ratio of that low intensity to the continuous background $E_0(1 + c\bar{\kappa}/\kappa_0)$ produced by the deep layers. For very large q the formula (7) of § 6, expressing the condition that the general absorption

may be neglected compared with the collision part of the monochromatic absorption, gives the values

$q = 10^6$	0.0148
10^7	0.00692
10^8	0.00321
10^9	0.00149
10^{10}	0.00069,

asymptotically coinciding with the values of the above tables.

8. To derive by means of these tables the contours of an absorption line it is now necessary to know the variation of α and q , *i.e.* of the complete monochromatic absorption coefficient s_0 in the vicinity of the wave-length λ_0 . A formula for this coefficient has been derived by W. Voigt.* His formula (34), by substituting $s_{0\rho} = \frac{4\pi}{\lambda_0} n\kappa$ and $\alpha\sqrt{1+y^2} = \alpha$, takes the form

$$s_{0\rho} = \frac{4\sqrt{\pi}\epsilon^2 N}{mc^3} \int_0^\infty ye^{-y^2} dy \arctan \frac{aby}{\mu^2 + \alpha^2 - b^2y^2},$$

where ϵ is the charge, m the mass of an electron, c the velocity of light, N the number of atoms producing the line per c.c., $b = 2\pi\nu_0 w/c^2$, $w^2 = 2kT/m_a$ (m_a is the mass of the atom, k the Boltzmann constant, w the mean Maxwellian velocity), $\mu = 2\pi(\nu - \nu_0)$. The quantity α is composed of two parts, $\alpha = \frac{1}{2}\nu' + 1/\tau$, the first representing the electromagnetic damping, the second the influence of collisions, τ being the mean free running time. For the first term the formula

$$\nu' = \frac{8\pi^2\epsilon^2\nu_0^2}{3mc^3},$$

used by R. Minkowski,† gives the value $\log \nu' = 8.143$. For the second term the formula $1/\tau = \pi n\sigma^2 v\sqrt{2}$, taking $T = 6000^\circ$, $n = 10^{12}p = 10^{14}$, $v(\text{electr.}) = 4 \times 10^7$, $\sigma = 10^{-8}$, gives 2×10^6 . So under the conditions existing in stellar atmospheres the collision term may be neglected compared with the other, and in Voigt's integral $\alpha = \frac{1}{2}\nu'$ is a constant. Computation shows that for hydrogen at 6000° $\log b = 11.193$, and for a difference $\nu - \nu_0$ corresponding to 1 Å. (at λ_{4000}) $\log \mu = 12.071$; hence the proper width of the line ν' is much smaller than the Doppler width b , while the latter is smaller than one angstrom. In evaluating the integral we neglect α^2 in the denominator, and put

$$y = \frac{\mu}{b} \tan \psi; \quad \frac{\nu'}{2\mu} \tan 2\psi = \tan \phi;$$

* W. Voigt, "Ueber das Gesetz der Intensitätsverteilung innerhalb der Linien eines Gasspektrums," *Sitz. Ber. München*, 1912, p. 603.

† *Zeitschr. für Physik*, **36**, 839 (1926).

then the integral takes the form

$$\int_0^{\infty} \phi y e^{-y^2} dy.$$

When y and $\tan \psi$ are increasing regularly from 0 to ∞ , 2ψ increases from 0 through $\frac{1}{2}\pi$ (for $y = \mu/b$) to π ; because $v'/2\mu$ is very small (except for very small μ) ϕ increases from 0 at first slowly, then from a small value jumps rapidly through $\frac{1}{2}\pi$ to nearly π , and remains so during the remaining region of integration. We therefore divide the integral into two parts by the limit $y = \mu/b$; in the first part we may put $\phi = v'/\mu$, in the second part $\phi = \pi$. The omitted contributions to the first integral by the part next to the upper limit $y = \mu/b$ are of the order of magnitude $\frac{v'}{\mu} e^{-(\mu/b)^2}$; moreover, they cancel out for the main part by the negative contributions $\phi - \pi$ near the lower limit of the second integral. Thus the integral becomes

$$\int_0^{\mu/b} \frac{v'}{\mu^2} y^2 e^{-y^2} dy + \pi \int_{\mu/b}^{\infty} y e^{-y^2} dy = -\frac{1}{2} \frac{v'}{\mu} e^{-\mu^2/b^2} + \frac{1}{2} \frac{bv'}{\mu^2} \int_0^{\mu/b} e^{-y^2} dy + \frac{\pi}{2} e^{-(\mu/b)^2}.$$

In the case of large μ/b the integral in the second term is nearly $\frac{1}{2}\sqrt{\pi}$; in this case the first term is extremely small and the whole expression may be written

$$\frac{1}{4} \sqrt{\pi} \frac{bv'}{\mu^2} + \frac{1}{2} \pi e^{-(\mu/b)^2}.$$

For decreasing μ the first term has to be multiplied by a factor which slowly decreases from unity and for $\mu = b$ reaches 0.84. For very small μ ($\mu < b$) the first of the two integrals may be developed

$$\frac{v'}{b^2 \mu} \left(\frac{1}{3} - \frac{1}{5} \frac{\mu^2}{b^2} + \dots \right).$$

For the centre of the line ($\mu = 0$) Voigt's integral takes the form

$$\int_0^{\infty} 2y e^{-y^2} dy \arctan \frac{2by}{v'}$$

which, by the same process of dividing, gives the result

$$\frac{1}{6} \left(\frac{v'}{b} \right)^2 + \frac{1}{2} \pi e^{-(v'/2b)^2}.$$

Since v'/b in all practical cases is small, this reduces to $\frac{1}{2}\pi$, the same result as is obtained by putting $\mu = 0$ in the general solution.

The result for the absorption coefficient is now

$$\begin{aligned} s_0 \rho &= \frac{2\sqrt{\pi} \epsilon^2 N}{mcb} \left\{ \frac{\sqrt{\pi} v' b}{2 \mu^2} + \pi e^{-(\mu/b)^2} \right\} & (\mu > b) \\ \text{or} & & \\ &= \frac{2\sqrt{\pi} \epsilon^2 N}{mcb} \left\{ \frac{2 v'}{3 b^2 \mu} + \pi e^{-(\mu/b)^2} \right\} & (\mu < b) \end{aligned} \quad (8)$$

It consists of two separate parts. One part is the Doppler broadened line, from the centre decreasing as a Gaussian curve with a modulus of width b , corresponding to the mean velocity of the atoms, equal to $\cdot 13\sqrt{\frac{1}{\mu_a} \frac{T}{6000}}$ angstrom (μ_a denotes the atomic weight). The factor b in the denominator shows that the maximum value of this term increases with decreasing width. The second part, the resonance wing of the line, proportional to $1/\Delta\lambda^2$, is identical with Unsöld's formula (1); it depends only on general physical constants and is the same for all kinds of atoms:

$$s_0\rho = \frac{2\pi\epsilon^4 N \lambda_0^2}{3m^2c^4 \Delta\lambda^2} \quad (9)$$

Introducing numerical values (valid for hydrogen and calcium atoms at 6000° and λ_{4000}) we have (numbers in brackets denote logarithms, $\Delta\lambda$ expressed in angstroms)

$$\left. \begin{aligned} \frac{s_0\rho}{N} &= [0.424 - 18]1/\Delta\lambda^2 \quad (\mu > b) \\ \text{or} \quad [0.935 - 16]\Delta\lambda \quad (\mu < b) \end{aligned} \right\} + [0.780 - 13]e^{-57\Delta\lambda^2} \quad (\text{for } H),$$

$$\left. \begin{aligned} \frac{s_0\rho}{N} &= [0.424 - 18]1/\Delta\lambda^2 \quad (\mu > b) \\ \text{or} \quad [0.338 - 13]\Delta\lambda \quad (\mu < b) \end{aligned} \right\} + [0.581 - 12]e^{-2280\Delta\lambda^2} \quad (\text{for } Ca).$$

A graphical representation of $\log s_0\rho/N$ (*vide* fig. 1, where each vertical division represents a factor 10) shows that in the outer parts the resonance term dominates and the Doppler term is imperceptible, while in the inner parts the latter dominates and the other is negligible, so that in each region simply one term has to be used, except for a narrow strip of transition, where the terms combine. This transition takes place at $\cdot 28$ A. for hydrogen, at $\cdot 07$ A. for calcium. It is clear from the above values that only an extremely small fraction of the emission of an atom is dispersed into the wings of a line.

9. We first consider the wings of strong lines, *i.e.* the cases where a moderate residual intensity is produced by the resonance part of the line-absorption curve. This residual intensity was found to depend solely on the quantity α ; for $r = 0.5$, according to Table II., we have $\log \alpha = -\cdot 35$ ($\alpha = \cdot 45$), hardly different for different assumptions regarding the constants of the atmospheric gases. Now $\alpha = (\kappa_0/s_0^2)(g/p')$ depends on the ratio of the general continuous absorption κ_0 and the monochromatic line absorption s_0 , combined with some gravity and pressure relations of the atmosphere, which were introduced because κ_0 depends on the electron pressure and s_0 does not.

The coefficient of general absorption has been the subject of the well-known investigations by Eddington* and Milne,† making use of Kramers's theoretical work on X-ray absorption. The aim of these

* *M.N.*, **84**, 104 (1924).

† *Ibid.*, **85**, 750, 768 (1925).

researches was the derivation of the mean absorption coefficient for the total radiation in the stellar interior; the same formulæ were then applied by Milne to the atmospheric layers. This general coefficient forms the continuous part of the total coefficient k , introduced in our formulæ, § 2, which determines the temperature gradient. For the line contours we want κ_0 , the continuous absorption coefficient for just the wave-length of the line considered. In his paper on

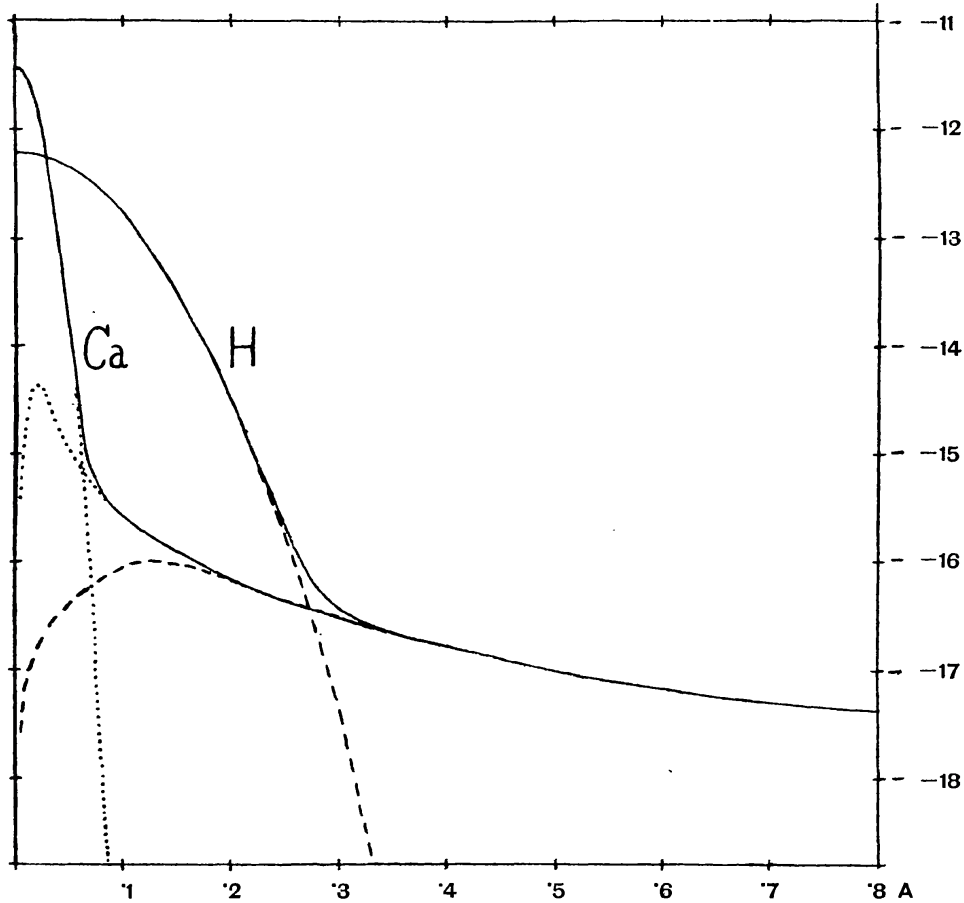


FIG. 1.

“Continuous Absorption,”* J. A. Gaunt has given a new derivation of the basic formulæ by wave mechanics. In the application of these formulæ to the case of a stellar atmosphere there are a number of uncertain factors; we discuss them in the Appendix (§ 13) to the present paper. The result may be put into the form

$$\kappa_0 = \frac{4\sqrt{6\pi}}{9} \frac{\epsilon^8 a'}{hc(mkT)^{3/2} \mu_a m_H} \frac{f(\nu\nu_1)}{\nu^3} e^{h\nu/kT} \quad (10)$$

where a' is the relative concentration in mass (fraction of the total mass) of the active atoms, μ_a is their atomic weight, and $f(\nu\nu_1)$ denotes

* *Phil. Trans.*, A, 229, 163 (1930).

the unimportant factor $\left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1}\right)$ with a numerical value of nearly 0.4.

Taking, according to that discussion, $a' = 1/200$, $\mu_a = 40$, we find for $T = 6000^\circ$ and wave-length 4000 Å. the numerical value $\log \kappa_0 = -3.8$.

For the resonance part of the line-absorption coefficient we found

$$s_0\rho = \frac{2\pi\epsilon^4}{3m^2c^4} N \left(\frac{\lambda_0}{\Delta\lambda}\right)^2.$$

Introducing n , the effective concentration of the atoms producing this line (fraction of the total mass), we may write $N = n\rho/\mu m_H$, and

$$s_0 = \frac{2\pi\epsilon^4}{3m^2c^4 m_H} \frac{n}{\mu} \left(\frac{\lambda_0}{\Delta\lambda}\right)^2 = [-1.00] \frac{n}{\mu} \left(\frac{\lambda_0}{\Delta\lambda}\right)^2.$$

Hence

$$\alpha = \frac{\sqrt{6\pi}}{\pi^2} \frac{m^{5/2} m_H c^4}{\epsilon^2 h k^{3/2}} \frac{a'}{\mu_a} \frac{g}{p' T^{3/2}} \lambda_0^3 f(\nu\nu_1) e^{c_2/\lambda_0 T} \left(\frac{\mu}{n}\right)^2 \left(\frac{\Delta\lambda}{\lambda_0}\right)^4. \quad (11)$$

In this expression the first two groups are numerical or physical constants; the quantities of the third group, depending on the constitution of the atmosphere, are rather uncertain and have to be assumed in some way; the next group consists of star parameters, whereas μ , n , and λ_0 are different for different absorption lines. It is the fractional concentration or abundance n which by means of this formula is determined from the observed $\Delta\lambda$.

In Unsöld's treatment of the problem the line width depends on the absolute number of absorbing atoms above the photosphere, and is used to derive this number, because the simplifying assumption is made of a continuously emitting black surface at the bottom with an overlying absorbing atmosphere. In reality the continuously emitting matter is composed of the very atoms that by their absorption produce the Fraunhofer lines, and all the atmospheric layers contribute to both the continuous emission and the line absorption. It is obvious that in such a case the strength of an absorption line does not depend on the number of active atoms themselves, but on their fractional concentration n relative to the total mass. The case is different for the centre of the line, where the residual intensity is nearly zero and depends on q , *i.e.* on s_0 . In this case the absorption takes place in the uppermost layers of the atmosphere; hence the case of a deeper continuously emitting source and another overlying absorbing atmosphere is realised here.

The half-width of a line is found by computing $\Delta\lambda$ for $\alpha = 0.45$. Then our formula (11) shows the following relations:—

- (1) For the same element $\Delta\lambda \sim \sqrt{n}$, where n is proportional to the weight of the transition producing the line. This corresponds to Unsöld's result that $\Delta\lambda \sim \sqrt{N}$.
- (2) For different elements in the same atmosphere $\Delta\lambda \sim \sqrt{n/\mu}$, where n/μ denotes the relative number of active atoms.

- (3) For giant stars with low g the lines are wider than for dwarf stars in the inverse ratio $\sqrt[4]{g}$.
- (4) For hotter stars the lines are wider than for cooler stars in the ratio $T^{3/8}$; the change of p' in the opposite direction does not compensate for this. This corresponds to Milne's result,* that by small gravity as well as by high temperature the width of an absorption line is increased.

We will now make use of the numerical values already computed for the case of the sun ($T = 6000^\circ$) and wave-length 4000 Å. :

$$\kappa_0 = [-3.8]; s_0 = [-1.00] \frac{n}{\mu} \left(\frac{\lambda_0}{\Delta\lambda} \right)^2 = [6.20] \frac{n}{\mu} \left(\frac{1}{\Delta\lambda} \right)^2 (\Delta\lambda \text{ in angstroms}).$$

For the sun $g = [4.44]$; for $p' = p/P$, which in the case of strong ionization approaches to 1, we take 2, assuming an equal number of atoms and electrons; hence $g/p' = [4.14]$. Then

$$\alpha = [-12.06] \left(\frac{\mu}{n} \right)^2 \Delta\lambda^4.$$

If for $\Delta\lambda$ we take the half-width of a line, where the residual intensity is 0.5, then $\alpha = 0.45$ ($\log = -0.35$), and we have

$$\frac{n}{\mu} = [-5.86] \Delta\lambda^2.$$

For the Ca^+ lines H and K the half-widths are 4.5 Å. and 6.5 Å., or $\Delta\lambda^2$ is 20 and 40; for all the Ca^+ atoms we put $\Delta\lambda^2 = 60$, and we find $n/\mu = [-4.08]$; $n = [-2.48]$. The Ca^+ atoms, therefore, constitute 1/300 in mass of the atmospheric gases of the sun.

For other lines with other wave-lengths the numerical constants are somewhat different. In κ_0 the different factors containing ν partly compensate one another, so that, going from 4000 Å. to 5890 Å. and 6560 Å. the value of $\log \kappa_0$ decreases by 0.50 and 0.58; the logarithms of the wave-length themselves increase by 0.168 and 0.215, so that the numerical coefficient in the formula for n/μ becomes smaller in the ratio $[0.25 + 0.34] = [0.59]$ and $[0.29 + 0.43] = [0.72]$. For the sodium lines D_1D_2 we read the half-widths from a diagram given by Unsöld, viz., 0.18 Å. and 0.29 Å., and for $H\alpha$ in the same way, 0.52 Å. †; from a table of Minnaert for the last-named the probably better value 0.64 Å. is found. ‡ Then for sodium and hydrogen we have

$$\frac{n}{\mu} = [-6.45](0.032 + 0.084) = [-7.39] \quad n = [-6.03] \quad (Na).$$

$$\frac{n}{\mu} = [-6.58](0.41) = [-6.97] \quad n = [-6.97] \quad (H).$$

For sodium this n denotes the concentration of neutral atoms; from the ionization formula the non-ionized fraction for $T = 6000^\circ$ and

* *M.N.*, **89**, 17, 157 (1928).

† *Zeitschr. für Physik*, **46**, 772 and 778.

‡ *Ibid.*, **45**, 613.

$P = 10^{-4}$ atm. is $[-2.6]$, so that the sodium atoms in total constitute $1/2500$ of the solar atmosphere. In this case, however, our treatment is not adequate, because the non-ionized fraction decreases regularly with decreasing electron pressure; hence s_0 in our formulæ should have been taken varying with P . For hydrogen $H\alpha$ comprises nearly 0.8 of all the transitions proceeding from the second quantum state; the concentration of hydrogen atoms in this state accordingly is given by $[-6.87]$. In the case of thermodynamic equilibrium the fraction of hydrogen atoms in the second state for 6000° is given by $[-8.5]$; since the total abundance of hydrogen atoms cannot exceed 1 , our result confirms the well-known phenomenon that the Boltzmann equilibrium formula gives too low numbers for the higher quantum states.

Unsöld has deduced the half-widths of several Fraunhofer lines from photometric measurements made at the Einsteinturm in Potsdam.* He did not publish these values themselves, but only the number of atoms NH deduced from them. They are proportional to our n , but for the wave-length factor in κ_0 . Comparing with our Ca^+ value and taking account of these factors we find the following mass-fractions:—

	$Na_0.$	$Al_0.$	$Ca^+.$	$Sr^+.$	$Ba^+.$	$Ca_0(4226).$	$Sr_0(4607).$
$10^6 n$.94	9.8	3300	2.6	.32	3.9	.009

All values deduced here depend on the value assumed for the continuous absorption coefficient and share its uncertainty. In § 13 reasons are given to expect that the coefficient is larger to an unknown amount, with an upper limit of 50 times. In this limiting case $\kappa_0 = [-2.1]$, and all the concentrations computed are increased 7 times, *i.e.* for Ca^+ we have $n = 1/43$ ($\log -1.63$), for Na $1/380$, for H (2nd state) $[-6.02]$.

10. We have now to derive the theoretical contours of an absorption line. As an example we take a line produced by Ca^+ by a percentage of $.0016$ (intermediate between the percentages producing the K and the H line). For each $\Delta\lambda$ we find $s_0\rho/N = s_0\mu m_H/n$ from formula (8), § 8. Then we compute $\alpha = \kappa_0 g/p's_0^2$. To derive the residual intensities r we make use of the tables of § 7, where the results for $c\bar{\kappa}/\kappa_0 = 1.5$ are used to construct a diagram, by means of which $r = f(\log \alpha)$ can be read.

The resulting contour curve is represented in fig. 2 † by the dotted line, which farther from the centre coincides with the dashed and the full line. It has a half-width of 5.3 A., between the values for H and K . It is funnel-shaped and has an intensity zero for $\Delta\lambda = 0$. It is entirely determined by the resonance part of the line-absorption coefficient. If now the percentage is diminished the line becomes narrower and its contour keeps the same figure if only the scale is widened proportionately to \sqrt{n} (and so the total quantity of energy lacking and the effective

* *Zeitschr. für Physik*, **46**, 772 (1928).

† If the continuous absorption coefficient should be increased 50 times, all the logarithmic concentrations indicated in this figure should be increased by $.8$; hence they would be -10 to -2 in fig. 2, -11.6 to -3.6 in fig. 3.

line width decrease with \sqrt{n} also). But now the centre between ± 0.1 A. becomes gradually distorted by the Doppler part of the line-absorption coefficient. In the figure the curve for $n = 10^{-4.8}$ is represented by the dashed line on a 10-times wider scale; hence this curve for the main part is identical with the first one, and only in the

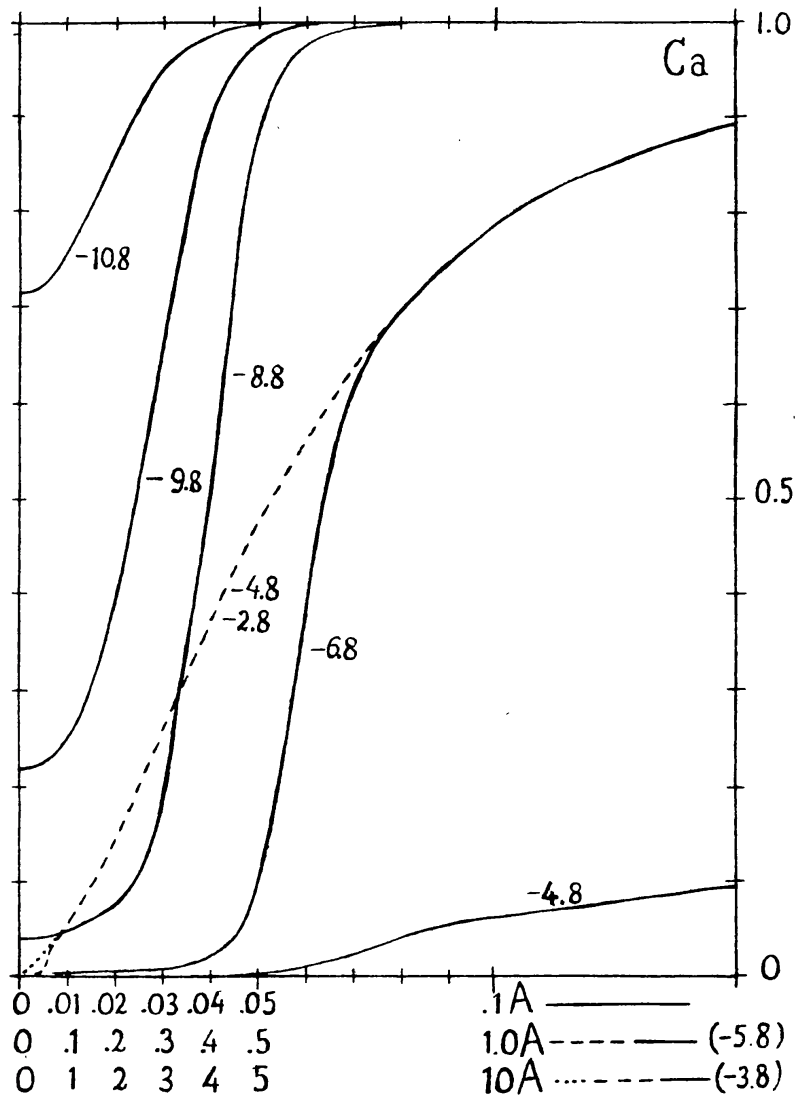


FIG. 2.

central part it is broadened and sack-shaped. For $n = 10^{-6.8}$ the intensity curve represented on a 100-times wider scale by a full line shows a much greater change; down to 0.08 A. with $r > 0.7$ the curves coincide in shape; then the narrow curve takes the form of a cup, at $\Delta\lambda = 0.05$ A. rapidly decreasing to nearly zero and remaining so. If now the percentage of active atoms is decreased still more (each curve is designated by the negative log of n), the absorption line is entirely determined by the Doppler curve; it contracts somewhat in

width, till at last it appears only $\cdot 05$ A. wide, and its central intensity increases, being $\cdot 04$ for $n = 10^{-8.8}$, $\cdot 21$ for $n = 10^{-9.8}$, $\cdot 71$ for $n = 10^{-10.8}$.

For other elements the contours have the same character and only the dimensions are somewhat different. With increasing atomic weight the Doppler curves contract in width; for higher temperatures they become wider. For the extreme case of hydrogen the contours

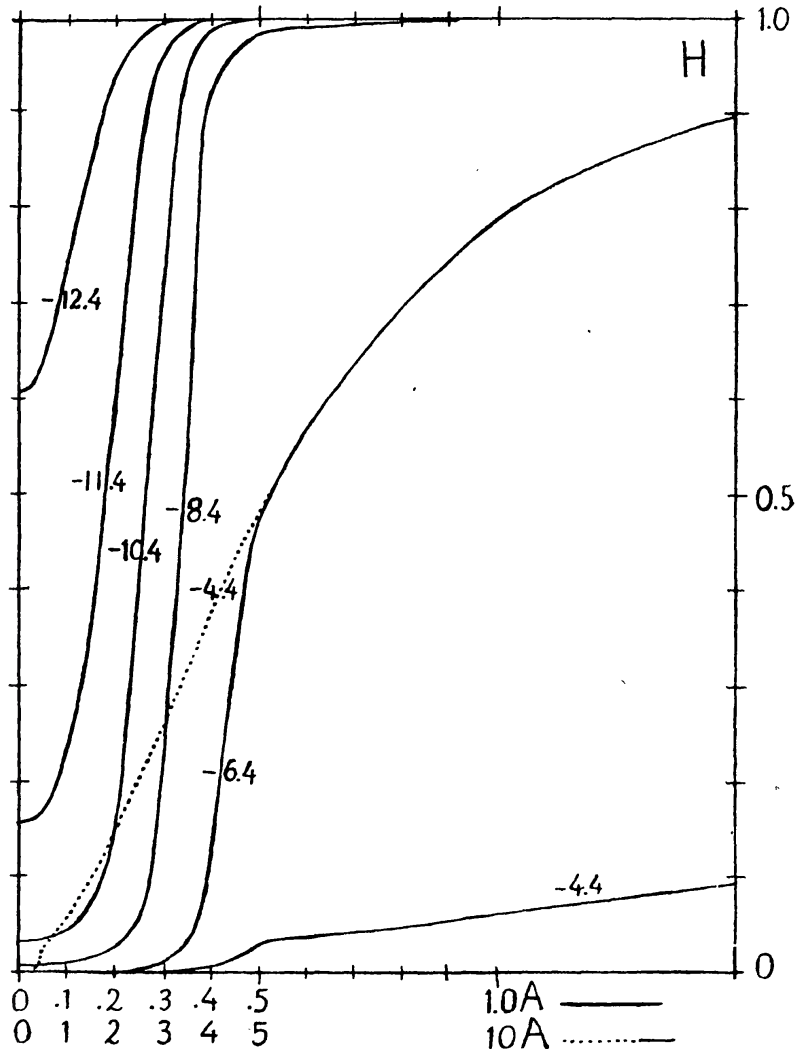


FIG. 3.

are represented in fig. 3; for identical wing width n is here $1/40$ of the calcium value. Here the cup-shaped central Doppler part appears at half-width $\Delta\lambda = \cdot 5$ A., contracts to $\cdot 3$, while the central intensity remains a few per cent. only, and then, while growing dimmer by increasing central intensity, keeps a width of $\cdot 3$ or $\cdot 4$ A.

These results are at variance with what is obtained by approximate methods. Unsöld in his first researches found the central intensity of an absorption line zero, because he only treated the wings by means of the resonance formula for the line-absorption coefficient which for

$\Delta\lambda = 0$ gives a coefficient ∞ . The central intensities observed are attributed by him to collisions of the radiating atoms, and he computes formulæ * for the residual intensity, depending on a quantity $\lambda = 2k/s$ in our notation. His conclusion that the central intensity for lines belonging to a multiplet must be the same, because k/s is the same for these lines, is not confirmed by our more exact computations. The divergence is not due to our general absorption k_0 , for our formulæ without α (§ 5, Table 1, and § 6) give the same result: the residual intensity is there a function of $q = \frac{p'}{g} s_0 Q$, which contains not only Q (which expresses the ratio of collision and radiation for these lines), but also s_0 , the total monochromatic absorption coefficient, which varies within a multiplet with the transition probability. It is easy to see that Unsöld's result follows from his assumption of one absorbing layer with definite k and s ; in the real case, however, of a gradual decrease of density with height, the strongest lines take their origin in the highest and thinnest layers where collisions are less important than in the deeper layers where the faint lines originate. Moreover, the validity of many of his conclusions is impaired if we are right in our inference that for moderate residual intensities the general absorption far exceeds the collision part of the line absorption.

The same holds for Woolley's computation of a mean pressure from the central intensity. The central intensity produced by the atmosphere does not agree with that produced by an average layer of constant density. Woolley tried to estimate the blackening influence of the higher layers, but he takes only account of the chromosphere and so finds their influence insignificant.

11. There is a marked difference between the observed contours of absorption lines and the theoretical results derived here on the basis of the assumptions of § 2. Whereas for strong lines we find the centre of the line practically dark, such a dark centre is an exception among the observed lines. The only case mentioned is the supergiant ϵ Aurigæ, where Elvey states that during the eclipse the intensity in $H\beta$ was zero in the centre (with a narrow emission line superposed).† On the drawing (p. 144 of his paper) $r = \cdot 15$ for $\Delta\lambda = \cdot 45$ Å. is the faintest point measured, and it is possible that the more central part has the cup-shape indicated for H in our diagram, fig. 3. Everywhere else we find central intensities from $\cdot 07$ (K line of Ca^+ in the sun),‡ up to some tenths, and even $> 0\cdot 5$ for broad shallow lines in several A and B stars. The often-quoted influence of collisions of atoms and electrons in the solar atmosphere is, as we have seen now, unable to explain these high central intensities, so we have to look for other influences that may change the contours of the lines. We may separate them into four kinds or groups:—

(1) Changes in the basic physical quantities determining the absorption.

* *Sommerfeld Festschrift*, p. 95 (1928).

† *Astroph. J.*, **71**, 140 (1930).

‡ Minnaert, *Zeitschr. für Physik*, **45**, 610.

- (2) Broadening of the absorption-coefficient curve.
 - (3) Broadening of the theoretical absorption-line contour by astronomical causes (size of the star, rotation).
 - (4) Change of the resulting contour by instrumental imperfections.
- These will now be considered in turn.

12. (1) The first group is named here only to indicate that the proper width of the line ν' , as we saw in § 8, will be increased for higher pressures by the influence of collisions. In such a case the resonance term in the formula for $s_0\rho$ increases, the same residual intensity in the wing corresponds to a lower percentage of active atoms, so that the maximum of the Doppler term is lowered, and the residual intensity in the centre is increased. A perceptible change for stellar absorption lines is not found on this account.

Holtmark* investigated the mutual resonance influence of atoms of the same kind and found an appreciable increase of the proper width ν' ; its amount is given by $1/4 \times 2.63 \times 10^7 \sqrt{N} r_0^{-3/2} \lambda / c$ (N , number of atoms per c.c.; r_0 , atomic diameter), which for $r_0 = 10^{-7}$, partial pressure = 10^{-6} atm., comes out 1.4×10^{10} , much larger than ν'_2 . Ornstein and Minnaert, however, found that this effect did not act as an increase of the proper width but as a broadening of the same kind as the Doppler effect. The quantities ν' and $1/\tau$ (§ 8) express a dissipation of energy, the basis of the absorption itself; since $\int s_0 d\lambda$ over the width of a line does not increase by this Holtmark effect, it belongs to the next group of influences.

(2) Among the causes that broaden the curve of the line absorption coefficient s_0 , we find in the first place the Stark effect, which has been shown by O. Struve† and A. Unsold‡ to be present in the A-type and B-type stars and in the sun. Because the atoms are vibrating in the rapidly changing electric fields of the surrounding ions and electrons the lines split up into a number of components, which by the continual variation of the fields are diluted into broad bands. Holtmark§ has computed the distribution of intensity over one of these bands on the supposition that the dimensions of the atoms are negligible compared with their distances apart. R. Gans|| afterwards repeated the computations for the case of appreciable atomic dimensions; for the densities occurring in stellar atmospheres he found that no perceptible corrections were necessary. For the case of surrounding point sources (ions and electrons) Holtmark finds the distribution of intensity as a function of the quantity $\Delta\nu/(2.61n^{2/3}\epsilon C)$ (n = number of point charges per c.c., ϵ their charge, C the constant of the Stark effect), and he gives a diagram in which it is represented by a curve. This curve in its main part resembles a Gaussian error curve, though in the extreme parts it does not fall off so rapidly as this curve; for practical computations we may use the expression $e^{-\Delta\nu^2/a^2}$, where $a = 1.96n^{2/3}\epsilon C$. If such a

* *Zeitschr. für Physik*, **34**, 722 (1925).

† *Astroph. J.*, **69**, 173; **70**, 85, 237 (1929).

‡ *Zeitschr. für Physik*, **46**, 779; **59**, 353.

§ *Ann. d. Phys.*, **58**, 577 (1919). || *Ibid.*, **66**, 396 (1921).

law of broadening is applied to a Doppler curve $e^{-(\mu/b)^2} = e^{-(2\pi\Delta\nu/b)^2}$ the result is a new Gaussian curve with exponent $\Delta\nu^2/(\alpha^2 + b^2/4\pi^2)$, while the maximum central amplitude is diminished in the ratio $b/2\pi\sqrt{\alpha^2 + b^2/4\pi^2}$. In our case, however, the matter is different. The light of central wave-length is coming from the highest atmospheric layers, where, owing to the small density, the Stark effect is absent; hence the top of the absorption-coefficient curve of fig. 1 is not widened at all; and the same holds for the whole Doppler part of the curve, for which the residual intensity is only a few hundredths. Only in the wings, where the moderate residual intensity is coming from deeper and denser layers, is the pressure broadening effective. The exact derivation of the line contour in the case of an appreciable Stark effect will be a complicated problem, because the different atmospheric layers, each with its own pressure effect, have to be combined. But so much we may say from our discussion, that in the centre of the line the residual intensity is *not* increased and only the wings are widened by this effect—at least, if the Stark effect in the wings is not so strong that they are diluted across the centre of the line. The total strength (effective width) of the line will then be increased. In the case of a very faint absorption line entirely produced by the Doppler part of the absorption, where the residual intensity in the centre is large already, the widening by the Stark effect will be accompanied by an increase of the central intensity, so that the contour is made shallower.

The Holtzmark effect, just mentioned, is also a pressure effect; its amount, as estimated above, comes out ten times smaller than b , so that in stellar atmospheres its influence probably is imperceptible.

Another cause of broadening of the absorption-coefficient curve would be given by irregular turbulent motions which were suggested by Unsöld* for the chromosphere. If the distribution of the velocities of these motions may be represented by an ordinary probability curve we will have a Doppler effect just as by the molecular translations but of a larger amount. Here we have a cause of broadening that affects the whole curve. If we represent it by $e^{-(2\pi\Delta\nu/\beta)^2}$, where $\beta = \nu_0 V/c$ (V = mean velocity), then the constant b in formula (8) for the line-absorption coefficient should be replaced by $\sqrt{b^2 + \beta^2}$, and the central top is lowered in the ratio $b/\sqrt{b^2 + \beta^2}$. Hence the central intensity in the absorption line is increased. The amount, however, is entirely insufficient; if this ratio, for example, is 1/10, so that the Doppler curve is ten times widened, the central intensity is displaced one number over our curves (figs. 2 and 3), increasing in the first numbers not at all, and then from 0.000 to 0.003, from 0.003 to 0.041, and from 0.041 to 0.214. Generally speaking, we cannot try to bring up the residual intensity in the centre to a moderate fraction by causes which broaden the absorption-coefficient curve, because then an impossibly large broadening would be required.

(3) The light emitted radially by a surface element of a regular

* *Astroph. J.*, **69**, 214 (1929).

stellar atmosphere, under the conditions stated in § 2, must exhibit absorption lines which are black in the centre, even if pressure and turbulence effects are taken into account. In the case of the Sun it is this radiation that can be investigated with the spectroscope. In the case of the stars we observe the total light of the star; central and border regions are superposed, and if there are differences in their spectrum the resulting image may become diffused and widened. The chief cause of widening is a rotation of the star. Its effect upon the contour of an absorption line was investigated by Carroll,* and by O. Struve and G. Shajn.† The radial velocity displacement is a linear function of the co-ordinate perpendicular to the projection of the rotation axis on the star disc; hence in the case of equal surface intensity the resulting contour of a line (of width 0) is given by a semi-circle, and in the case of a darkening towards the limb it can easily be found by integrating the intensity along a strip parallel to the axis. This cause of broadening, because it works upon the absorption line itself, is more effective in raising the central intensity than the preceding group of causes which worked upon the absorption coefficient curve. Elvey ‡ could explain in this way the wide diffuse lines in a number of A-type and B-type stars, where the residual intensity in the centre reaches 0.60 to 0.90. In computing the rotational velocity, however, he assumes that in stars without rotation the central intensity of the line is as high as 0.30 to 0.40. These intensities have to be explained in another way.

(4) In stellar spectra the last-named group of causes, the lack of sufficient resolution and definition, may change the apparent contours of a line in a considerable degree. When a three-prism spectrograph is used, producing a scale of 10 Å. to 1 mm. in the middle parts, the projected slit width usually corresponds to .2 or .3 Å., which by diffraction is still further increased. Such narrow black central depressions as are represented in fig. 2 for *Ca* (which may be taken as a specimen for all the heavier elements), with a width of .10 to .05 Å., cannot be distinguished in these spectra. They are diffused and the black centre is effaced. Computing the apparent central intensity in the case in which the light is diffused evenly over .3 Å., we find from the curves of fig. 2 (and others interpolated between them),

$$\begin{array}{cccc} \text{for } n = & 10^{-4.8} & 10^{-5.4} & 10^{-6.2} & 10^{-6.8}; \\ r = & .04 & .08 & .30 & .48. \end{array}$$

In all these cases the theoretical intensity in the centre is $< .01$. For still lower concentrations, where the real width of the line is small and the wings have disappeared, the central intensity of the broadened line rises nearer to 1. The total absorption of the line, or the equivalent width, is not changed by this instrumental widening, and may now be used to determine n . These considerations show the importance of taking stellar spectra with a much greater resolving power for the purpose of an exact determination of line contours.

* *M.N.*, **88**, 548 (1928).

† *M.N.*, **89**, 222 (1929).

‡ *Astroph. J.*, **71**, 221 (1930).

This explanation of course holds only for some of the observed lines in spectra. In the case of the hydrogen lines the black Doppler nucleus has a width of 0.5 to 1 Å., and cannot be diffused by the same instrumental causes. Still Elvey, in his three-prism spectra,* finds central intensities of .19 and .14 for Sirius ($H\beta$ and $H\gamma$), .22 and .16 for α Lyræ, .13 for Procyon, and for most stars much larger values. In the case of the Sun we can free ourselves from the rotational as well as (for a large part) the instrumental widening by taking the spectrum of one part of the disc only and making use of a large dispersion. For $H\alpha$ Minnaert finds a central intensity .15; and for the K line .07—smaller than other observers have found (Unsöld gives .46 for $H\alpha$, .16 for K); this may be an indication that large errors, especially by stray light which is very difficult to remove, may spoil such measures. If we take Minnaert's results as right, we have here a discrepancy between theory and observation which cannot be explained by any of the causes mentioned above. The irregular figure of the contour of the solar K line, giving rise to the reversal K_2 and the double reversal K_3 , may perhaps be considered as an indication that reversals of temperature, or more generally irregular deviations from the adopted temperature and pressure laws in the atmosphere, are responsible for the observed central intensity.

13. *Appendix. The Continuous Absorption in the Atmospheric Layers of a Star.*—The formulæ derived by Gaunt give the absorption exerted on a stream of radiation by the interaction of an electron and a positively charged nucleus. If the energy E before absorption was positive, we have the case of a collision of a free electron which by absorption increases its energy. If E before absorption was negative, the electron was bound to the nucleus and we have the case of ionization by absorption of radiation. The formulæ hold strictly for the case of both energies as well as ν approximating to 0. For the case of finite ν with $E \rightarrow 0$ (+ or -), as well as for the case of finite ν and E (negative) with final $E' = 0$, a development in series showed that only multiplying factors g had to be added, not much different from 1 (1.2, 1.1, 0.9, 0.8).

The formula for positive E (free-free transitions) can be applied easily to the case of hyperbolic orbits around ionized atoms, which then may be considered as nuclei with charge $Z = 1$. It is not at once applicable to electrons penetrating into the atom and during part of their orbit subject to the stronger attraction of a higher charged centre. Their contribution to the absorption will increase the amount computed in the first-named case by an unknown factor.

The absorption coefficient for the case of free electrons with energy E and velocity v , as given by Gaunt's formula (6.10), identical with the formula derived formerly by Kramers by means of the correspondence principle, is

$$a_0(E, \nu) = \frac{4\pi Z^2 \epsilon^6}{3\sqrt{3} h c m^2 v^3}$$

* The slit width is not given, but probably it was not much above .3 Å. for $H\gamma$.

for one atom per unit density of electrons. We take $Z = 1$. The average of $1/v$ for a Maxwellian distribution of velocities is given by

$$\overline{1/v} = \sqrt{\frac{2m}{\pi kT}}.$$

For the electron density we introduce its value P/kT , where P is the electron pressure. Denoting by a the fraction of the total mass which is formed by the active atoms, and by F the unknown factor just alluded to, we find the mass-absorption coefficient by dividing by $m_H \mu (= [221 - 24]\mu)$; the result is

$$k_0(\nu) = \frac{4\sqrt{6\pi}}{9} \frac{\epsilon^6 P \cdot a \cdot F}{hc(mkT)^{3/2} \nu^3 m_H \mu}.$$

Introducing numerical values for the constants for the case of $T = 6000^\circ$, $\lambda = 4000 \text{ \AA}$., we find

$$k_0(\nu) = [0.889 - 3] PaF/\mu.$$

The formula for negative E (bound electrons thrown out by the ionization process) as given by Gaunt (6.21) is

$$a_1(E, \nu) = \frac{1}{n^2} \frac{16\pi^2}{3\sqrt{3}} \frac{Z^2 \epsilon^6}{ch^3 \nu^3} \Delta\nu,$$

where

$$\Delta\nu = \frac{4\pi^2 m Z^2 \epsilon^4}{h^3 n^3}.$$

Here again the ionized atom may be considered as a nucleus with charge $Z = 1$. Introducing the values of the constants (again taking 4000 \AA . and 6000°) this amounts to $[834 - 16]/n^5$, and the mass-absorption coefficient from this source becomes

$$k_1(E, \nu) = [0.613 + 8]/n^5 \mu.$$

If the resulting energy $E' = 0$, this absorption would be concentrated in a number of separate lines ν_n (for $n = 1, 2, 3, 4 \dots$), each of which collects the energy within a region $\Delta\nu$ (which contains the factor $2/n^3$, for small n to be replaced, after Milne, by $\frac{1}{(n-1/2)^2} - \frac{1}{(n+1/2)^2}$). For another E' all these ν_n are replaced by somewhat larger values, $\nu = \nu_n + E'/h$; thus each ν_n is extended into a continuous band, reaching from ν_n as its lower sharp edge towards ∞ . To compute the absorption coefficient for a certain ν we have to add the contributions of all bands which have their edges at a lower $\nu_n < \nu$; the strength of each band depends on the relative number of atoms in the corresponding state, for which we take the Boltzmann formula; hence

$$a_1(\nu) = \frac{1}{\nu^3} \frac{16\pi^2 \epsilon^6}{3\sqrt{3} ch^3} \sum_{(n)} \frac{1}{n^2} e^{-h(\nu_1 - \nu_n)/kT} \Delta\nu.$$

This value behaves very irregularly (*vide* fig. 4), decreasing slowly and then jumping up to a higher value at each band edge. In order to have an expression independent of the chance values of the excitation energies, we will for the summation substitute an integral, assuming a quasi-infinite number of band edges, each of which has $d\nu$ instead of the finite $\Delta\nu$. In the simplified case represented by the Gaunt formula we may put $n^2/1 = h\nu_1/h\nu_n$; then the integral becomes

$$\int_0^\nu \frac{\nu_n}{\nu_1} e^{-h(\nu_1 - \nu_n)/kT} d\nu_n = \frac{kT}{h} \left\{ \left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1} \right) e^{-h(\nu_1 - \nu)/kT} + \frac{kT}{h\nu_1} e^{-h\nu_1/kT} \right\}$$

The last term is small compared with the others, and may be omitted. The result is that the factor $\Delta\nu$ ($= [0.818 + 15]Z^2/n^3$) is replaced by $\frac{kT}{h}$ ($= [0.99 + 14]$, so that the constant in k_1 becomes $[6.883]$), multiplied by a mainly exponential factor depending on the ionization

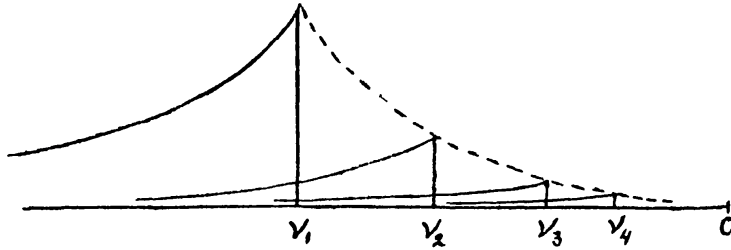


FIG. 4.

potential. We can express all the energies by corresponding potentials ($h\nu = \epsilon V/300$, $kT = \epsilon V_T/300$); then the expression above becomes

$$\frac{V - V_T}{V_1} e^{-(V_1 - V)/V_T}.$$

Here V_1 is the ionization potential of the element. For wave-length 4000 Å., $V = 3.1$; for $T = 6000^\circ$, $V_T = 0.52$; the absorption coefficient per atom in this case will be

$$[6.883] \frac{2.6}{V_1} e^{-(V_1 - 3.1)/0.52}.$$

For atoms with a high ionization potential this exponential form is very small; only atoms with a low ionization potential will give an appreciable contribution to this absorption coefficient. These atoms, however, are to a large part ionized, and only a small neutral fraction is able to take part in this absorbing process. From the ionization formula, expressed in the quantities above, viz.

$$\frac{x_1 P}{x_0} = 10^{-0.5T + 2.5} e^{-V_1/V_T} \quad (P \text{ in absolute units})$$

we derive for the cases of strong ionization, when x_1 is nearly 1,

$$x_0 = 10^{0.5T - 2.5} e^{+V_1/V_T} P.$$

If we multiply the computed absorption coefficient by this fraction we get the average absorption coefficient per atom of this element present in the atmosphere (all states of ionization counted together). In this product

$$[7.38] \frac{V - V_T}{V_1} T^{-2.5} e^{V/V_T P},$$

the ionization potential has disappeared (but for the unimportant factor $(V - V_T)/V_1$), whereas the electron pressure appears as a factor of this part of the absorption coefficient too. Hence this expression is the same for the different elements of low ionization potential and each contributes according to its concentration in the stellar atmosphere. This holds as long as the ionized fraction remains above, say, $1/2$ (for $T = 6000^\circ$ and $P = 10^{-4}$ atmospheres this corresponds to an ionization potential of 8 and comprises *K, Na, Ca, Sr, Ti, Cr*, perhaps *Fe*); for other elements the neutral fraction increases very slowly towards 1, and the product of this fraction with the absorption coefficient decreases rapidly with increasing ionization potential.

Restoring the analytical expressions for $a_1(\nu)$ and for $\frac{x_1}{x_0}$, viz.

$$a_1(\nu) = \frac{16\pi^2}{3\sqrt{3}} \frac{\epsilon^6}{ch^3\nu^3} \frac{kT}{h} \left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1} \right) e^{h\nu/kT - h\nu_1/kT}$$

$$\frac{x_1}{x_0} P = \frac{(2\pi m)^{3/2}}{h^3} (kT)^{5/2} e^{-h\nu_1/kT},$$

multiplying with a factor a' (denoting the concentration of the relevant elements), and dividing by $m_H \mu$, to have the mass-absorption coefficient we find

$$k_1(\nu) = \frac{4\sqrt{6}\pi}{9} \frac{\epsilon^6 P a'}{hc(mkT)^{3/2} \nu^3 m_H \mu} \left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1} \right) e^{h\nu/kT}.$$

This expression is the same as our expression for $k_0(\nu)$, only with the addition of two factors at the end. In both a factor indicating the concentration of relevant atoms occurs. The factor a in $k_0(\nu)$ is the fraction of the atoms ionized; the most numerous atoms, those of hydrogen (and helium) are not ionized, and among the less numerous metal atoms we may suppose half of them ionized. The fraction a' in $k_1(\nu)$ indicates the abundance of atoms of those elements that are at least half ionized. These fractions will be of the same order of magnitude. The additional factor in $k_1(\nu)$ amounts to $[2.59 - 0.4]$; hence it appears that the ionization part of the absorption coefficient comes out 150 times larger than the collision part, at least if the latter, by the unknown factor F introduced in the beginning, is not increased in the same ratio. If this is not the case, we may to a first approximation neglect the collision part in comparison with the other one. Introducing numerical values we have then

$$k_1(\nu) = [0.89 - 3][2.59 - 0.4] \frac{P a'}{\mu}.$$

As to the concentration a' we can at present only make a rough guess. We will assume that the metal atoms are 1 per cent. of the total mass, and that one half of them contribute to the formula. Hence taking $a' = 1/200$, and μ , their mean atomic weight = 40, we find

$$k(\nu) = \kappa_0 P = [-3.8]P.$$

This value has been used in the computations of § 9. Probably, however, it is too low, because the unknown factor F in k_0 is underrated here. We can indicate an upper limit for it. Treating the electrons penetrating into the atoms as meeting a single nucleus with $Z = 20$, the coefficient of absorption from free-free transitions would become 400 times larger. Still more important is the contribution of the hydrogen atoms, if we may treat the electrons penetrating into them as if they meet single nuclei with charge $Z = 1$ and $\mu = 1$. Because their abundance is nearly 1, they would produce in this case a coefficient $200 \times 40 = 8000$ times higher than the ionized metal atoms do. This is an upper limit for F ; it means that the total k , in which in this case free-free transitions play the dominant part, may be at most 50 times the value given above, *i.e.* $\kappa_0 < [-2.1]$. The great abundance of hydrogen atoms in stellar atmospheres makes it highly desirable that the absorption caused by free-free transitions of electrons along the nucleus of a neutral hydrogen atom should be investigated more exactly; this appears to be the chief agent determining the transparency of such atmospheres. Since we cannot make as yet any estimate of its real value, we have not changed the above value in the computations of § 9.

14. We can also use these results to derive the mean absorption coefficient of such an atmosphere for the total radiation. The net stream of all wave-lengths which passes through the atmosphere is a given quantity; hence the net stream equation for a single ν :

$$\frac{dy(\nu)}{\rho dh} = k(\nu)z \quad \text{or} \quad \frac{1}{k(\nu)} \frac{dy(\nu)}{\rho dh} = z(\nu)$$

has to be written for the total radiation

$$\sum_{\nu} z(\nu) = \sum_{\nu} \frac{1}{k(\nu)} \frac{dy(\nu)}{\rho dh} = \frac{\bar{1}}{k(\nu)} \frac{d}{\rho dh} \sum y(\nu).$$

It is not the simple mean of the absorption coefficients which we want, but their harmonic mean; it is not the absorption but the transparency that is averaged.

In the case of an absorption coefficient produced by free-free transitions the derivation of this $k = 1/\frac{\bar{1}}{k(\nu)}$ is easy. The only variable factor in $k(\nu)$ is $1/\nu^3$; and we have

$$\int \nu^3 E_{\nu}^{\dagger} d\nu / \int E d\nu = \int \frac{\nu^6 d\nu}{e^{h\nu/kT} - 1} / \int \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1} = \left(\frac{kT}{h}\right)^3 \int \frac{x^6 dx}{e^x - 1} / \int \frac{x^3 dx}{e^x - 1}.$$

The second integral is $\frac{1}{18}\pi^4 = 6.49$; the first can be written

$$6!(1 + 2^{-7} + 3^{-7} + \dots) = 726.01;$$

so that instead of $1/\nu^3$ in the $k(\nu)$ we have $\left(\frac{h}{kT}\right)^3 \frac{6.49}{726.01}$ in the average.

To obtain the general absorption coefficient we have to multiply the coefficient, computed for ν , by the factor

$$\frac{1}{111.80} \left(\frac{h\nu}{kT}\right)^3.$$

In our case ($T = 6000^\circ$, ν for 4000 Å.) $h\nu/kT = 5.97$ and the factor amounts to $213/112 = 1.91$. Milne, in computing the simple mean, had to multiply $k(\nu)$ by $\frac{15}{\pi^4} \left(\frac{h\nu}{kT}\right)^3$, which gives a result 17 times greater than the harmonic mean.

In the case of $k(\nu)$ produced by ionization the matter is much more difficult. Here a number of bands are superposed, each with a sharp edge at the lower side, so that the transparency is largest just below these edges. It is clear that the average transparency will depend in a high degree on the number and the distribution of these bands, *i.e.* on the different ionization potentials. The procedure of smoothing them into an infinite number of band edges cannot be followed here, because then just the most transparent parts would be obliterated. In the formula

$$\text{harm. mean } k = \int \mathbf{E}(\nu) d\nu \int \frac{\mathbf{E}(\nu)}{k(\nu)} d\nu$$

the latter integral breaks up into a number of separate terms, between the limits $\nu_1, \nu_2, \nu_3 \dots$, the ionization band edges. We omit first the common factor in $k(\nu)$,

$$\frac{64\pi^4 Z^4 \epsilon^{10} m}{3\sqrt{3} ch^6} \quad (\text{in which } Z \text{ shall be taken } = 1),$$

and preserve only the variable factor $\frac{e^{-h(\nu_1 - \nu_n)/kT}}{\nu^3 n^5}$. In the approximation of Gaunt's deduction, $1/n^2$ represents $h\nu_n/h\nu_1$, so that n^5 in these integrals may be replaced by 1, $(\nu_1/\nu_2)^{5/2}$, $(\nu_1/\nu_3)^{5/2}$, etc. If it is permissible to neglect in each interval between two edges all the more remote fainter bands and to preserve only the next one, we have for this latter integral

$$\int_{\nu_1}^{\infty} \mathbf{E}\nu^3 d\nu + \left(\frac{\nu_1}{\nu_2}\right)^{5/2} e^{(h\nu_1 - \nu_2)/kT} \int_{\nu_2}^{\nu_1} \mathbf{E}\nu^3 d\nu + \left(\frac{\nu_1}{\nu_3}\right)^{5/2} e^{(h\nu_1 - \nu_3)/kT} \int_{\nu_3}^{\nu_2} \mathbf{E}\nu^3 d\nu + \dots$$

The arbitrary character of this value in consequence of our ignorance as to the chance values of these separate ν_n can be avoided in the following way:—

If the discontinuous factors $\left(\frac{1}{\nu_n}\right)^{5/2} e^{-h\nu_n/kT}$ before the integrals are

replaced by the continuously changing values $\left(\frac{1}{\nu}\right)^{5/2} e^{-h\nu/kT}$, the integration can be extended from 0 to ∞ without knowledge of the band edges. The integral is then too small because for the integrand is taken a continuous curve through the minimum values at the darkest edges, and the real values from these points rapidly increase to the transparent parts close to the next edges. For the factor f , by which it has to be multiplied, we take the mean ratio between the real, increasing values and the minimum edge value within each interval. The minimum integral is given by

$$\nu_1^{5/2} e^{h\nu_1/kT} \int_0^\infty \frac{\nu^{7/2}}{e^{h\nu/kT}-1} e^{-h\nu/kT} d\nu = \nu_1^{5/2} e^{h\nu_1/kT} \left(\frac{kT}{h}\right)^{9/2} \int_0^\infty \frac{x^{7/2}}{e^x-1} e^{-x} dx$$

for which the numerical value

$$\cdot 64 \nu_1^{5/2} e^{h\nu_1/kT} \left(\frac{kT}{h}\right)^{9/2}$$

can be found. For the factor f , by which it has to be multiplied,

$$f = \frac{1}{x_n - x_{n+1}} \int_{x_{n+1}}^{x_n} \left(\frac{x}{x_{n+1}}\right)^{5/2} e^{x-x_{n+1}} dx,$$

cannot be given an exact value, because it depends on the unknown quantities x_n and $\Delta x = x_n - x_{n+1}$. To avoid mathematical difficulties it was evaluated, after substituting the exponents 3 and 2 for 5/2, for different values of x_n and Δx . Since in the minimum integral only the contributions of x between $x = 1$ and $x = 5$ are relevant, we want the factor f only for x_n between these limits. Here f increases from nearly 1.2 to nearly 8, if Δx increases from .5 to 2. It is not probable that for such x_n intervals as large as 2 in the values of $h\nu/kT$ between successive band edges will occur, especially because a number of elements with ionization potential below 7 or 8 take part in the absorption, and each produces a number of band edges. It is not probable, therefore, that the uncertainty reaches this factor 8, though it is difficult to ascertain a trustworthy lower limit.

For the harmonic mean of k , without the constant factors, and without the concentration factor for the neutral atoms taking part in the process, we now have

$$\frac{\pi^4 (kT)^4}{15 \left(\frac{h}{kT}\right)^4} \cdot 64f \cdot \left(\frac{h\nu_1}{kT}\right)^{5/2} \left(\frac{kT}{h}\right)^7 e^{h\nu_1/kT} = \frac{6 \cdot 49 \left(\frac{h}{kT}\right)^3 \left(\frac{h\nu_1}{kT}\right)^{-5/2}}{\cdot 64f} e^{-h\nu_1/kT}.$$

The non-ionized fraction of the atoms is given by

$$x_0 = (2\pi m)^{-3/2} h^3 (kT)^{-5/2} e^{h\nu_1/kT} P,$$

and with a' again denoting the abundance of the atoms concerned and μ_a their atomic weight, the total formula for the continuous absorption is found :

$$k = \frac{16\pi^6 \sqrt{2\pi}}{45 \sqrt{3}} \cdot \frac{\epsilon^{10}}{cm^{\frac{1}{2}} m_H} (kT)^{-5\frac{1}{2}} \left(\frac{kT}{h\nu_1}\right)^{5/2} \frac{1}{\cdot 64f} \cdot \frac{a'}{\mu} P.$$

Introducing numerical values, taking $h\nu_1/kT = 6.1 \text{ Volt}/.52 \text{ Volt} = 12$ for the average atoms, and assuming as above $a' = 1/200$, $\mu_a = 40$, we find $k = [-3.7]_f^I P$.

In this formula the coefficient is proportional to the $-5\frac{1}{2}$ th power of the temperature, whereas in Kramers's formula the power is $-4\frac{1}{2}$. There is, however, an additional factor containing T . Now with increasing temperature, more metals with higher ionization potentials take a perceptible part in this absorption; hence the average ν_1 increases with T , and the factor $kT/h\nu_1$ certainly is not proportional to T . It may, however, remove part of the power $-5\frac{1}{2}$ in the preceding factor.

The Resonance Theory of the Origin of the Moon (Second Paper).
By Harold Jeffreys.

1. It was shown by Sir G. H. Darwin that if the earth and moon were formerly united as a single fluid mass with their present total angular momentum, the period of rotation would be about 4 hours, so that the period of the semi-diurnal tide raised by the sun at any given place would be about 2 hours. This is approximately the period of the slowest free oscillation of a fluid mass of the earth's mean density, and Darwin suggested in consequence that the sun's tides would have been enormously magnified by resonance. The principle is that if a system is able to oscillate, and disturbing forces are so timed that in whatever direction their point of application is moving of its own accord we give it a push in the direction of motion, an indefinitely large amplitude may be ultimately worked up. If the mass became drawn out into a sufficiently long ellipsoid it would become unstable, and the suggestion was that at this stage a portion broke off to form the moon.

It has always been recognized that the validity of this theory depends on the smallness of friction; for friction necessarily fixes an upper limit to the amplitude of an oscillation attainable by resonance. In an earlier paper * I showed that the requisite coincidence of periods can be attained only if we allow for the earth's lack of homogeneity; the central core must be considered. This introduces a possibility of friction in the tidal currents in the outer shell flowing over the central core. This friction can now be estimated; and it turns out to be sufficient to invalidate the theory.

2. A complete solution of the problem would be very laborious, but fortunately turns out to be unnecessary. We notice first that if we have a system specified by a differential equation of the type

$$\ddot{x} + n^2x = f \sin nt \quad . \quad . \quad . \quad (1)$$

* *M.N.R.A.S.*, **78**, 116-131, 1917.