**Astrophysics.** — *The Central Intensity in the Fraunhofer Lines.* By A. PANNEKOEK.

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1. In our paper "The Theoretical Contours of Absorption Lines"[1]) it was stated that there is a marked contradiction between the observed contours of absorption lines and the theoretical results derived on the basis of the most plausible assumptions on stellar atmospheres. Whereas theory demands an intensity zero (i.e. below one percent) in the centre of a line, the observed central intensities are from 0.07 upward to several tenths. The different causes for broadening, considered in that paper, cannot explain why for the bulk of the Fraunhofer lines in the solar spectrum the central intensity should not be practically zero. The explanation by means of collisions, given by UNSÖLD and used also by WOOLLEY, rests on the use of approximate formulae, which assume a constant density throughout the absorbing layers. Owing to the very high value of the monochromatic absorption coefficient in the centre of a line only the upper layers of the atmosphere, where collisions do not play any part, determine the central intensity.

An explanation may be found, however, if in some way light from the wings could be transferred to the centre. In the wings the intensity has all values between the full background brightness and zero; the atoms producing the line absorb this winglight, are lifted to a higher energy level and must in falling back re-emit this energy. In the existing discussions, however, it was always assumed that the excited atom re-emits radiation of the same wave-length as had previously been absorbed, just as if it remembered by what means it came into the higher level. So the equations for each wave-length, used in deriving the resulting intensities, were quite independent of other wave lengths.

Recently a mutual influence of different wave lengths in the realm of a line has been inferred by O. HALPERN[2]) from the classical formulae of resonance. If the incident light of a frequency $\nu$ somewhat deviating from the proper frequency of the resonator $\nu_0$, consists of single finite

[1]) Monthly Notices R. A. S. **91**, p. 139 (1930).

[2]) Zeitschr. f. Physik **67**, S. 523 (1931).

or damped wave-trains, it can be resolved by FOURIER analysis into a broadened band, decreasing with increasing wave-length difference; each separate wave-length in this band produces forced vibrations of the resonator, and it appears that the intensity of these induced vibrations has two marked maxima, one for frequency $\nu$ and another for $\nu_0$. In a different way R. VAN DER RIET WOOLLEY [1]) came to the same result, by making use of the model description of the resonance broadening of the lines, assuming the energy levels themselves broadened. We have to assume in this model that within a broadened level (here called a state of the atom) the weight of each special level depends on $\triangle\nu$ in the same way as the absorption and diffusion coefficient in ordinary theory depends on $\triangle\nu$, i.e. shows the same strong core and the same wings, decreasing with $1/\triangle\nu^2$. Then from each level in the lower state I the probability of being lifted to some level of the higher state II is proportional to the product of their weights. A beam of light of central frequency $\nu_0$ is absorbed practically only by the mean lower level $I_0$, from which the atoms are lifted to the mean higher level $II_0$; they return mostly to $I_0$, emitting $\nu_0$, and for a small part to deviating lower levels $I_1$, emitting wing frequencies $\nu_1$. A beam of light of wing frequency $\nu_1$ is absorbed with a smaller chance by $I_0$, lifting the atoms to the deviating higher level $II_1$, and is absorbed with the same chance by $I_{-1}$ lifting these atoms to the mean level $II_0$. For both cases the atoms return with dominant chance to $I_0$; in the first case $\nu_1$ is emitted, in the second case $\nu_0$. Hence radiation of some wing frequency after absorption is emitted half as radiation of the same frequency and half as radiation of normal frequency. At the same time of the radiation absorbed of normal frequency the greater part is emitted in the same frequency and a small part is distributed over the wing frequencies.

Quantum mechanics assigns different resonance coefficients to the different states of an atom; each of them is proportional to the sum total of the transition probabilities from this state, i.e. inverse proportional to its average life time. Then the absorption coefficient for a wing frequency is determined by the resonance coefficients of the lower and the upper state, added together, in accordance with WEISSKOPF and WIGNER's result. The absorbed energy is divided into two unequal parts; the part emitted as normal frequency is produced by the wings of the lower state only, and the part re-emitted in the same frequency depends on the wings of the upper state.

It seemed worth while to consider the bearing of these theoretical ideas upon the explanation of the central intensity in the Fraunhofer lines. In view of the complications introduced by the mutual influence of the different radiations we are obliged to assume a simplified model of the mechanism of formation of the line. We cannot make use of the

[1]) Monthly Notices R. A. S. **91**, p. 977 (1931).

exact formulae, which take account of the variations in density and in absorption coefficient with depth, because they can be only solved by numerical integrations. We will assume an atmosphere of constant density, where the coefficients of absorption and diffusion have the same ratio in all the layers; moreover we neglect the increase of the absorption coefficient in the centre of the line due to collisions of excited atoms. For the case of each wave-length independent of the others we have then the formulae given by MILNE[1]). We have to modify them according to our case of mutual influence.

2 We assume the core of the absorption line to have a certain width $w$ and a constant high diffusion coefficient $s_0$. For the wings we have in the two states much smaller coefficients $s_1, s_2$, decreasing with increasing distance to the centre. After a wave-length $\nu_0$ of the core has been absorbed by a layer of atoms, this energy is distributed in the re-emission over the whole width of the line, each wave-length taking part in ratio of its $s$. When a wave-length of the wing has been absorbed ($s = s_1 + s_2$), the energy is emitted partly as light of the same wave-length ($s_2$), partly ($s_1$) as light of the central core, evenly distributed over the width of the core. The coefficient of general absorption is $k$: the intensities of the outward and inward streams of radiation are called $I_0$ and $I'_0$ for a wave-length of the core, $I_1$ and $I'_1$ (functions of $\lambda$) for a wave-length of the wing. The total amount of absorbed energy of the central wave-lengths in the layer $dx$ is $(s_0 + k)(I_0 + I'_0)\, w\, dx$; the part with $k$ is transformed into heat; the rest is distributed over the line in such a way that each receives

$$\frac{s_0 \text{ or } s_1}{s_0 w + \int s_1 d\lambda}\, s_0 (I_0 + I'_0)\, w\, dx.$$

The energy of wing light absorbed in this layer amounts to

$$dx \int (s + k)(I_1 + I'_1)\, d\lambda,$$

of which the $k$ part again is transformed into heat. From the rest each wave-length in the core receives $\dfrac{dx}{w} \displaystyle\int s_1 (I_1 + I'_1)\, d\lambda$, and each wave-length in the wing receives from its own absorbed energy $s_2 (I_1 + I'_1)$. Hence the equations for the streams of energy are ($x$ counted downward)

[1]) Monthly Notices R. A. S. **89**, p. 7

$$\left.\begin{aligned}
\frac{dI_0}{dx} &= +(s_0+k)I_0 - \frac{1}{2w}\int s_1(I_1+I'_1)\,d\lambda - \tfrac{1}{2}\frac{s_0}{s_0 w + \int s_1 d\lambda} s_0 w (I_0+I'_0) - kE \\
\frac{dI'_0}{dx} &= -(s_0+k)I'_0 + \frac{1}{2w}\int s_1(I_1+I'_1)\,d\lambda + \tfrac{1}{2}\frac{s_0}{s_0 w + \int s_1 d\lambda} s_0 w (I_0+I'_0) + kE \\
\frac{dI_1}{dx} &= +(s+k)I_1 - \tfrac{1}{2}s_2(I_1+I'_1) - \tfrac{1}{2}\frac{s_1}{s_0 w + \int s_1 d\lambda} s_0 w (I_0+I'_0) - kE \\
\frac{dI'_1}{dx} &= -(s+k)I'_1 + \tfrac{1}{2}s_2(I_1+I'_1) + \tfrac{1}{2}\frac{s_1}{s_0 w + \int s_1 d\lambda} s_0 w (I_0+I'_0) + kE
\end{aligned}\right\} \quad (1)$$

As in our former papers we put $I + I' = y$ (mean stream density), $I - I' = z$ (net outward stream); by adding and subtracting we find

$$\frac{dy_0}{dx} = (s_0+k)\,z_0 \quad ; \quad \frac{dy_1}{dx} = (s+k)\,z_1\,; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\left.\begin{aligned}
\frac{dz_0}{dx} &= (s_0+k)\,y_0 - \frac{1}{w}\int s_1 y_1 d\lambda - \frac{s_0}{s_0 w + \int s_1 d\lambda} s_0 w\, y_0 - 2kE \\
\frac{dz_1}{dx} &= (s+k)\,y_1 - s_2\, y_1 - \frac{s_1}{s_0 w + \int s_1 d\lambda} s_0 w\, y_0 - 2kE
\end{aligned}\right\} \quad . \quad . \quad (3)$$

Multiplying the first of these equations (3) with $w$, integrating the second after $\lambda$, and adding and subtracting the results, we have

$$\left.\begin{aligned}
\frac{d}{dx}\left(wz_0 + \int z_1\, d\lambda\right) &= k\left(wy_0 + \int y_1\, d\lambda\right) - 2kE\left(\int d\lambda + w\right) \\
\frac{d}{dx}\left(wz_0 - \int z_1\, d\lambda\right) &= \left(\frac{2\int s_1 d\lambda}{s_0 w + \int s_1 d\lambda} s_0 + k\right) wy_0 - \int (2s_1+k)\, y_1 d\lambda + \\
&\quad + 2kE\left(\int d\lambda - w\right)
\end{aligned}\right\} \quad . \quad (4)$$

We put

$$\left.\begin{array}{l}\dfrac{\int s_1 d\lambda}{s_0 w + \int s_1 d\lambda} s_0 = S \\[2ex] u = (2S+k)\, w y_0 - \int (2s_1 + k)\, y_1\, d\lambda \\[2ex] v = k w y_0 + k \int y_1\, d\lambda \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then we have

$$\frac{du}{dx} = (2S+k)(s_0+k)\, w z_0 - \int (s+k)(2s_1+k)\, z_1\, d\lambda$$

$$\frac{dv}{dx} = k(s_0+k)\, w z_0 + k \int (s+k)\, z_1\, d\lambda$$

Again differentiating and substituting (4) and (5) we find

$$\left.\begin{array}{l}\dfrac{d^2u}{dx^2} = a^2\left(u + 2kE\left(\int d\lambda - w\right)\right) + b^2\left(v - 2kE\left(\int d\lambda + w\right)\right) \\[2ex] \dfrac{d^2v}{dx^2} = c^2\left(u + 2kE\left(\int d\lambda - w\right)\right) + e^2\left(v - 2kE\left(\int d\lambda + w\right)\right)\end{array}\right\} . \quad (6)$$

where

$$\left.\begin{array}{ll} a^2 = \tfrac{1}{2}(2S+k)(s_0+k) + \tfrac{1}{2}\,\overline{(s+k)(2s_1+k)} & c^2 = \tfrac{1}{2}k(s_0+k) - \tfrac{1}{2}k\,\overline{(s+k)} \\ b^2 = \tfrac{1}{2}(2S+k)(s_0+k) - \tfrac{1}{2}\,\overline{(s+k)(2s_1+k)} & e^2 = \tfrac{1}{2}k(s_0+k) + \tfrac{1}{2}k\,\overline{(s+k)} \end{array}\right\} \quad (7)$$

and the mean values are defined by

$$\overline{s+k} \int z_1\, d\lambda = \int (s+k)\, z_1\, d\lambda \quad ; \quad \overline{(s+k)(2s_1+k)} \int z_1\, d\lambda = \int (s+k)(2s_1+k)\, z_1\, d\lambda.$$

The values of the net stream $z_1$ which act as the weights in averaging $s$ are variable with the depth; hence in different layers the averaging is made with different weights and the resulting mean values are functions of the depth (with increasing depth the weight of the high values of $s$ becomes smaller). We may, however, neglect this variability, because it affects only small secondary terms in the coefficients. If we treat $a^2\, b^2\, c^2\, e^2$ as constant coefficients, the solution of the equations (6) presents no difficulty. Eliminating $v$ we have

$$\left.\begin{array}{l}\dfrac{d^4u}{dx^4} - (a^2+e^2)\dfrac{d^2u}{dx^2} + (a^2e^2 - b^2c^2)\left(u + 2kE\left(\int d\lambda - w\right)\right) + \\[2ex] \qquad + \left((b^2-a^2)\int d\lambda + (b^2+a^2)\, w\right) 2k \dfrac{d^2E}{dx^2} = 0\end{array}\right\} . \quad (8)$$

If we put

$$a^2+e^2=p^2+q^2 \qquad a^2e^2-b^2c^2=p^2q^2. \quad . \quad . \quad . \quad . \quad (9)$$

the solution is

$$\left.\begin{aligned} u = {} & Ae^{-px}+A'e^{px}+Ce^{-qx}+C'e^{qx}- \\ & -\frac{1}{(D^2-p^2)(D^2-q^2)}\left\{p^2q^2\left(\int d\lambda-w\right)2\,kE+\right. \\ & \left.\qquad\qquad +\left((b^2-a^2)\int d\lambda+(b^2+a^2)\,w\right)2\,k\frac{d^2E}{dx^2}\right\} \end{aligned}\right\} . \quad (10)$$

It is easily seen that by the condition of finite radiation at great depths the coefficients $A'$ and $C'$ must vanish. For the particular solutions we have

$$\frac{1}{(D^2-p^2)(D^2-q^2)}E=\frac{-1}{p(p^2-q^2)}\left\{e^{px}\int_x^\infty Ee^{-px}\,dx+e^{-px}\int_0^x Ee^{px}\,dx\right\}+$$

$$+\frac{1}{q(p^2-q^2)}\left\{e^{qx}\int_x^\infty Ee^{-qx}\,dx+e^{-qx}\int_0^x Ee^{qx}\,dx\right\}$$

$$\frac{1}{(D^2-p^2)(D^2-q^2)}\frac{d^2E}{dx^2}=\frac{-1}{p(p^2-q^2)}\left\{e^{px}\int_x^\infty \frac{d^2E}{dx^2}e^{-px}\,dx+e^{-px}\int_0^x \frac{d^2E}{dx^2}e^{px}\,dx\right\}+$$

$$+\frac{1}{q(p^2-q^2)}\left\{e^{qx}\int_x^\infty \frac{d^2E}{dx^2}e^{-qx}\,dx+e^{-qx}\int_0^x \frac{d^2E}{dx^2}e^{qx}\,dx\right\}=$$

$$=\frac{-1}{p(p^2-q^2)}\left\{(pE_0-E'_0)\,e^{-px}+p^2e^{px}\int_x^\infty Ee^{-px}\,dx+p^2e^{-px}\int_0^x Ee^{px}\,dx\right\}$$

$$+\frac{1}{q(p^2-q^2)}\left\{(qE_0-E'_0)\,e^{-qx}+q^2e^{qx}\int_x^\infty Ee^{-qx}\,dx+q^2e^{-qx}\int_0^x Ee^{qx}\,dx\right\},$$

where $E_0$ and $E'_0$ have been written for the values of $E$ and $\frac{dE}{dx}$ at the surface. Putting now

$$\frac{p^2q^2}{p^2-q^2}\left(\int d\lambda-w\right)=c_1 \qquad \frac{1}{p^2-q^2}\left\{(b^2-a^2)\int d\lambda+(b^2+a^2)\,w\right\}=c_2. \quad (11)$$

we have

$$\left.\begin{aligned}
u &= Ae^{-px} + Ce^{-qx} + \\
&+ 2k\left[\left(\frac{c_1}{p^2}+c_2\right)\left(pe^{px}\int_x^\infty Ee^{-px}dx + pe^{-px}\int_0^x Ee^{px}dx\right) + \right. \\
&\qquad\qquad \left. + c_2\left(E_0 - \frac{E'_0}{p}\right)e^{-px}\right] \\
&\qquad - \text{the same form with } q \text{ for } p. \\
\frac{du}{dx} &= -pAe^{-px} - qCe^{-qx} + \\
&+ 2k\left[\left(\frac{c_1}{p^2}+c_2\right)\left(p^2e^{px}\int_x^\infty Ee^{-px}dx - p^2e^{-px}\int_0^x Ee^{px}dx\right) - \right. \\
&\qquad\qquad \left. - pc_2\left(E_0 - \frac{E'_0}{p}\right)e^{-px}\right] \\
&\qquad - \text{the same form with } q \text{ for } p. \\
\frac{d^2u}{dx^2} &= +p^2Ae^{-px} + q^2Ce^{-qx} + \\
&+ 2k\left[\left(\frac{c_1}{p^2}+c_2\right)\left(p^3e^{px}\int_x^\infty Ee^{-px}dx + p^3e^{-px}\int_0^x Ee^{px}dx - 2p^2E\right) + \right. \\
&\qquad\qquad \left. + p^2c_2\left(E_0 - \frac{E'_0}{p}\right)e^{-px}\right] \\
&\qquad - \text{the same form with } q \text{ for } p.
\end{aligned}\right\} \quad . \quad (12)$$

The first of the equations (6) now gives the value of $v$

$$v = \frac{1}{b^2}\frac{d^2u}{dx^2} - \frac{a^2}{b^2}u + 2kE\left(\frac{b^2-a^2}{b^2}\int d\lambda + \frac{b^2+a^2}{b^2}w\right)$$

Since

$$\frac{1}{b^2}\left\{-2k\left(\frac{c_1}{p^2}+c_2\right)2p^2E + 2k\left(\frac{c_1}{q^2}+c_2\right)2q^2E\right\} =$$

$$= -2kE\left(\frac{b^2-a^2}{b^2}\int d\lambda + \frac{b^2+a^2}{b^2}w\right)$$

the terms with $2E$ vanish and we find

$$\left.\begin{aligned}
v &= \frac{p^2-a^2}{b^2} A e^{-px} + \frac{q^2-a^2}{b^2} C e^{-qx} + \\
&+ 2k \frac{p^2-a^2}{b^2} \left[ \left( \frac{c_1}{p^2} + c_2 \right) \left( p e^{px} \int_x^\infty E e^{-px} dx + p e^{-px} \int_0^x E e^{px} dx \right) + \right. \\
&\qquad\qquad\qquad\qquad \left. + c_2 \left( E_0 - \frac{E'_0}{p} \right) e^{-px} \right] \\
&\qquad \text{— the same form with } q \text{ for } p\,; \\
\frac{dv}{dx} &= -p \frac{p^2-a^2}{b^2} A e^{-px} - q \frac{q^2-a^2}{b^2} C e^{-qx} + \\
&+ 2k \frac{p^2-a^2}{b^2} \left[ \left( \frac{c_1}{p^2} + c_2 \right) \left( p^2 e^{px} \int_x^\infty e^{-px} dx - p^2 e^{-px} \int_0^x E e^{px} dx \right) - \right. \\
&\qquad\qquad\qquad\qquad \left. - p c_2 \left( E_0 - \frac{E'_0}{p} \right) e^{-px} \right] \\
&\qquad \text{— the same form with } q \text{ for } p.
\end{aligned}\right\} \quad (13)$$

The boundary conditions in the deep interior are satisfied by the vanishing of the terms with $E$. The boundary conditions at the surface determine the integration constants $A$ and $C$. At the surface (where the quantities shall be denoted by the index 0) we have

$$y_{00} = z_{00} \quad , \quad y_{10} = z_{10} \quad ; \text{ or } \quad y_{00} - \frac{1}{s_0+k} \frac{dy_{00}}{dx} = 0 \quad ; \quad y_{10} - \frac{1}{s+k} \frac{dy_{10}}{dx} = 0$$

for each wave-length; for their assemblage we have

$$(s_0 + k)\, w y_{00} - w \frac{dy_{00}}{dx} = 0 \quad ; \quad \overline{s+k} \int y_{10}\, d\lambda - \int \frac{dy_{10}}{dx}\, d\lambda = 0 \ . \quad (14)$$

Since

$$u_0 = (2\,S + k)\, w y_{00} - \overline{2s_1 + k} \int y_{10}\, d\lambda \quad ; \quad v_0 = k w y_{00} + k \int y_{10}\, d\lambda$$

we find by solving

$$\left.\begin{aligned}
N w y_{00} &= k u_0 + (\overline{2s_1 + k})\, v_0 \quad ; \quad N w \frac{dy_{00}}{dx} = k \frac{du_0}{dx} + \overline{2s_1 + k} \frac{dv_0}{dx} \\
N \int y_{10}\, d\lambda &= -k u_0 + (2S + k)\, v_0 \,; \; N \int \frac{dy_{10}}{dx} d\lambda = -k \frac{du_0}{dx} + (2S+k) \frac{dv_0}{dx} \\
N &= k\,(2S + k + \overline{2s_1 + k})
\end{aligned}\right\} \quad (15)$$

The mean values $\overline{2s_1+k}$ occurring in these equations are constant surface values; the weights are here the surface values $y_{10}=z_{10}$, the observed intensities.

The surface values of $u$ and $v$ and their derivates, which must be introduced here, are found by taking $x=0$ in the equations (12) and (13). By means of the quantities

$$\left.\begin{aligned}
&\left(\frac{c_1}{p^2}+c_2\right)\int_0^\infty 2k\,E\,e^{-px}\,p\,dx=P \quad ; \quad c_2\left(E_0-\frac{1}{p}\,E'_0\right)=R\\
&\left(\frac{c_1}{q^2}+c_2\right)\int_0^\infty 2k\,E\,e^{-qx}\,q\,dx=Q \quad ; \quad c_2\left(E_0-\frac{1}{q}\,E'_0\right)=T
\end{aligned}\right\} \qquad (16)$$

we can express them:

$$\left.\begin{aligned}
u_0&=(A+R+P)+(C-T-Q)\\
\frac{du_0}{dx}&=p\,(-A-R+P)+q\,(-C+T-Q)\\
v_0&=\frac{p^2-a^2}{b^2}\,(A+R+P)+\frac{q^2-a^2}{b^2}\,(C-T-Q)\\
\frac{dv_0}{dx}&=\frac{p^2-a^2}{b^2}\,p\,(-A-R+P)+\frac{q^2-a^2}{b^2}\,q(-C+T-Q)
\end{aligned}\right\} \quad . \quad . \quad (17)$$

By introducing the quantities:

$$\left.\begin{aligned}
\alpha_0&=k+(2s_1+k)\,\frac{p^2-a^2}{b^2} \quad ; \quad \beta_0=k+(2s_1+k)\,\frac{q^2-a^2}{b^2}\\
\alpha_1&=-k+(2S+k)\,\frac{p^2-a^2}{b^2} \quad ; \quad \beta_1=-k+(2S+k)\,\frac{q^2-a^2}{b^2}
\end{aligned}\right\} \quad . \quad (18)$$

the surface conditions take the form

$$\left.\begin{aligned}
&\alpha_0\,\{(A+R)(s_0+k+p)+P(s_0+k-p)\}+\\
&\qquad\qquad +\beta_0\,\{(C-T)(s_0+k+q)-Q(s_0+k-q)\}=0\\
&\alpha_1\,\{(A+R)(s+k+p)+P(s+k-p)\}+\\
&\qquad\qquad +\beta_1\,\{(C-T)(s+k+q)-Q(s+k-q)\}=0
\end{aligned}\right\} \quad . \quad (19)$$

and the surface values of $y$, the emitted intensities, are given by

$$\left.\begin{aligned}
&Nwy_{00}=\alpha_0\,(A+R+P)+\beta_0\,(C-T-Q)\\
&N\int y_{10}\,d\lambda=\alpha_1\,(A+R+P)+\beta_1\,(C-T-Q)
\end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

It appears that in eliminating the arbitrary constants $A$ and $C$ by means of (19) the quantities $R$ and $T$ are eliminated at the same time. We find:

$$\left.\begin{aligned} w y_{00} &= \frac{1}{ND}(\alpha_0\beta_1-\alpha_1\beta_0)\{2p\alpha_0 P(s+k+q)-2q\beta_0 Q(s+k+p)\} \\ \int y_{10}d\lambda &= \frac{1}{ND}(\alpha_0\beta_1-\alpha_1\beta_0)\{2p\alpha_1 P(s_0+k+q)-2q\beta_1 Q(s_0+k+p)\} \\ D &= \alpha_0\beta_1(s_0+k+p)(s+k+q)-\alpha_1\beta_0(s+k+p)(s_0+k+q). \end{aligned}\right\} \quad (21)$$

In the computation of the integrals $P$ and $Q$ we assume that the function $E$ is determined by the general absorption coefficient $k$ only; we have then $E=E_0(1+kcx)$, where c is a numerical coefficient of the order of unity depending on the wave-length. We find:

$$\left.\begin{aligned} P &= \left(\frac{c_1}{p^2}+c_2\right)\int_0^\infty 2kE_0(1+kcx)\,e^{-px}p\,dx \\ &= \frac{1}{p^2-q^2}\left((b^2-a^2+q^2)\int d\lambda+(b^2+a^2-q^2)\,w\right)2kE_0\left(1+\frac{kc}{p}\right) \\ Q &= \frac{1}{p^2-q^2}\left((b^2-a^2+p^2)\int d\lambda+(b^2+a^2-p^2)\,w\right)2kE_0\left(1+\frac{kc}{q}\right) \end{aligned}\right\} \quad (22)$$

3. For numerical evaluation of these expressions it is first necessary to consider the different orders of magnitude of the coefficients. The coefficient $s_0$ is very large, $10^5$ or $10^6$ times larger than $s$ in the wings and $k$. If we consider $k$ and $s$ to be of order zero, and $s_0$ of order 2, then $S$ is of order 1. In the expressions (7) for $a^2, b^2, c^2, e^2$ the second terms may be neglected, and from

$$p^2=a^2+e^2=(S+k)(s_0+k) \quad ; \quad q^2=\frac{a^2e^2-b^2c^2}{a^2+e^2}=k(s+k); \quad . \quad (23)$$

we see that $p$ is of the order $1^1/_2$, $q$ of the order 0. Thus it appears, that the uncertainty due to the variability of the mean values, is removed for the greater part and remains only in the value of $q$. From the expressions, deduced now from (18)

$$\alpha_0=k\left(1+\frac{2s_1+k}{2S+k}\right) \quad ; \quad \alpha_1=-k\frac{s+k}{s_0} \quad ; \quad \beta_0=-2s_1 \quad ; \quad \beta_1=-2(S+k) \quad (24)$$

we find their order of magnitude 0, $-2$, 0, and 1. Hence the products $\alpha_0\beta_1$ and $\alpha_1\beta_0$ are of the order 1 and $-2$, and in the expression for $D$ the first term has the order 3, the second the order $1^1/_2$. So in $D$ we may restrict ourselves to the first term only, and in the other equations

(21) the value $\alpha_1\beta_0$ may be omitted. The result for $y_{00}$ now assumes the much simpler form

$$wy_{00}=\frac{2p}{(2S+k)(s_0+k+p)}P+\frac{2q\,.\,2s_1\,.\,(\overline{s+k}+p)}{k(2S+k)(s_0+k+p)(\overline{s+k}+q)}Q$$

The integrals $P$ and $Q$, which by these substitutions take the form

$$\left.\begin{aligned}P&=\left\{\frac{2S+k}{S+k}w-\frac{2s_1(\overline{s+k})}{(S+k)(s_0+k)}\int d\lambda\left\{2kE_0\left(1+\frac{kc}{p}\right)\right.\right.\\Q&=\left\{\frac{S}{S+k}w+\int d\lambda-\frac{2s_1(\overline{s+k})}{(S+k)(s_0+k)}\int d\lambda\left\{2kE_0\left(1+\frac{kc}{q}\right)\right.\right.\end{aligned}\right\}\quad . \tag{25}$$

can be simplified in the same manner. Since $\frac{1}{w}\int s_1\,d\lambda$ is nearly equal to $S$, the first term in the brackets is the most important in $P$, whereas in $Q$ it is just the second term which is one order larger than the first. Substituting these chief terms only into the equation for $y_{00}$ we find:

$$\left.\begin{aligned}y_{00}&=\frac{2pk}{(S+k)(s_0+k+p)}2E_0\left(1+\frac{kc}{p}\right)+\\&\quad+\frac{2q(\overline{s+k}+p)}{(2S+k)(\overline{s+k}+q)}\frac{\int 2s_1d\lambda}{w(s_0+k+p)}2E_0\left(1+\frac{kc}{q}\right)\end{aligned}\right\} \tag{26}$$

This equation may be used to compute the value of the radiated intensity of the centre of a Fraunhofer line; the background intensity is $E_0(1+c)$. To find the order of magnitude of these terms and to see in what way the determining quantities play a part in them, we simplify them by omitting all minor terms. We substitute

$$p=\sqrt{Ss_0}\quad;\quad S=\frac{\int s_1\,d\lambda}{w}\quad;\quad q=\sqrt{k(\overline{s+k})}.$$

In the second term for $q/(s+k+q)$ we write $1\Big/\left(1+\sqrt{\frac{s+k}{k}}\right)$; the factor $1+\frac{kc}{q}$ for $c=1$ is equal to $\left(1+\sqrt{\frac{k}{s+k}}\right)$, so that combined they give a factor $\sqrt{\frac{k}{s+k}}$, which for $k>s$ approaches to 1.

The simplified formula is now

$$y_{00}=\left(\frac{2k}{\sqrt{Ss_0}}+2\sqrt{\frac{S}{s_0}}\sqrt{\frac{k}{s+k}}\right)2E_0$$

If we compare it with the intensity for the case that the centre of the line is not influenced by the wings [1]), viz (for $c = 1$)

$$y'_{00} = 2E_0 \sqrt{\frac{k}{s_0}} \quad : \quad y'_{10} = 2E_0 \sqrt{\frac{k}{s+k}},$$

we see that the first value, if we identify it with the first term above, is diminished, but that a second larger term is added, produced by the radiation coming from the wings. We may state the result for its value in this way, that for $k$ in the ordinary expression now $S = \frac{1}{w}\int s_1 d\lambda$ is substituted, which represents the emission of wing light (of intensity $y'_{10}$) distributed over the width of the core. In ordinary theory the intensity of the centre is produced by black radiation throwing light into this centre (coefficient $k$), which is weakened by the atomic diffusion (coefficient $s_0$). Now, that $k$ is very small, the light thrown into this centre by the wings (coefficient $S$) takes its place.

We will apply these formulas to the case of $Ca$ atoms, for which numerical data are given elsewhere [2]). The theoretical absorption coefficient according to these data may be expressed by

$$s = 1/\triangle\lambda^2 + [6.16]\, e^{-(48\triangle\lambda)^2},$$

where the second term denotes the DOPPLER curve of the core and the first term the wings. There is of course something arbitrary in the separation of wings and core; strictly speaking each partial wave-length interchanges with every other wave-length. There is, however, a sudden transition from the low wing values to the steep and high central values at nearly $\triangle\lambda = {}^1/_{16}$ A. So we take here our boundaries, at a half-width 0.063 A. The average value of $s$ within these limits $s_0 = [5.63]$. The value of the wing integral up from this lower limit $\int s d\lambda = \int d\lambda/\triangle\lambda^2 = 16$. If we assume $s_1 = {}^1/_2 s$ we have $S = 128$.

Then the additional term in the central intensity, without the last factor which for dark wings falls considerably below 1, is 0.034, expressed in fraction of the background. This factor may be evaluated for different values of $k$ (increasing values of $k$ mean in reality decreasing values of $s$ and $s_0$, i.e. of the concentration). From equation (26) we see that the average value is determined by the integral

$$\int \frac{q}{s+k+q}\left(1+\frac{k}{q}\right) 2s_1\, d\lambda = \int 2s_1 \sqrt{\frac{k}{s+k}}\, d\lambda =$$

$$= \sqrt{\frac{k}{4s'}}\, ln \frac{\sqrt{k+s'}+\sqrt{s'}}{\sqrt{k+s'}-\sqrt{s'}} \times \int 2s_1\, d\lambda$$

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[1]) Handbuch der Astrophysik III 1, S. 305; MILNE, Monthly Notices R. A. S. **89**, p. 7.

[2]) Monthly Notices R. A. S. **91**, p. 151. The values in square brackets are logarithms.

where $s'$ denotes the value of $s$ at the lower boundary 0.063 A. For $k = 1$, 10, 100 (when the ordinary intensity would be .0015, .005, .015) the factor is .22, .46, .78, and the additional term is .007, .016, .027.

The values found here for the second term, show that it is not sufficient in itself to explain the observed central intensities of spectral lines. It is, however, important that the central values from a few thousandths, given by the former treatment, now rise above one hundredth, so that additional influences, not strong enough by themselves, may raise them somewhat higher. Moreover the computation has been made with the classical resonance formula; according to the quantum mechanics the classical coefficient of the resonance term should be multiplied by factors $f$, certainly amounting in some cases to 4 and perhaps higher still [1]); in this case the additional term in the centre has to be multiplied by $\sqrt{f}$. It must be remarked, however, that the resonance coefficient for the lowest state of an atom is practically zero; so for the lines absorbed by this lowest state $s_1 = 0$ and the central intensity is not increased by radiation from the wings.

[1]) MINNAERT u MULDERS, Zeitschr. f. Astrophysik **2**, S. 179; PANNEKOEK, These Proceedings **34**, p. 763.