

## THE HYDROGEN LINES NEAR THE BALMER LIMIT.

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1. It is well known that in celestial spectra only a limited number of hydrogen lines of the Balmer series is visible; near the series limit they merge and at last are effaced, a continuous spectrum making its appearance a good deal on the long wave-length side of the theoretical limit. In the spectrum of ordinary A-type stars not more than 20 lines can be distinguished; Menzel gives 29 lines (up to H<sub>31</sub>) for the chromospheric spectrum.

The disappearance of the higher members of the Balmer series is not exclusively due to their merging; other influences are at work. H. P. Robertson and Jane M. Dewey\* have treated the disturbance of the higher electron orbits by the electric fields due to free electrons or ions coming near to the atom. The disturbed orbits cannot be quantized, and the transition to or from the second level takes place in the form of continuous absorption or emission. Thus line radiation is changed into continuous radiation.

The theory of this effect is treated in the same way as the theory of the Stark effect. We introduce parabolic co-ordinates  $\xi$  and  $\eta$ , defining a point in a meridian plane containing the direction of the electric field by the parameters of two parabolas intersecting at that point. The 3rd co-ordinate is the equatorial angular co-ordinate  $\phi$ . Then we have the quantum conditions

$$\left. \begin{aligned} \int p_\phi d\phi &= 2\pi p_\phi = n_3 h, \\ \int p_\xi d\xi &= \int \frac{d\xi}{\xi} \sqrt{-p_\phi^2 + 2\alpha_1 \xi + 2m\bar{W}\xi^2 - mEF\xi^3} = n_1 h, \\ \int p_\eta d\eta &= \int \frac{d\eta}{\eta} \sqrt{-p_\phi^2 + 2\alpha_2 \eta + 2m\bar{W}\eta^2 + mEF\eta^3} = n_2 h, \end{aligned} \right\} \quad (1)$$

where  $\alpha_1 + \alpha_2 = 2me^2$ ,  $\bar{W}$  is the (negative) total energy of the electron, and  $F$  is the field strength. In order to reduce these expressions to dimensionless numerical values, we express the co-ordinates  $\xi$ ,  $\eta$  in terms of the radius  $a_0$  of the lowest orbit as unit. Then replacing  $\alpha_1$  and  $\alpha_2$  by  $2\gamma_1 me^2$  and  $2\gamma_2 me^2$  ( $\gamma_1 + \gamma_2 = 1$ ), and considering that  $p_\phi^2 = n_3^2 h^2 / 4\pi^2$ ,  $me^2 a_0 = h^2 / 4\pi^2$ ,  $2m\bar{W}a_0^2 = -h^2 / 4\pi^2 n^2$ , and  $mea_0^3 = (a_0^2/e)h^2 / 4\pi^2$ , which we denote by  $Eh^2 / 4\pi^2$  ( $E = 5.846 \times 10^{-8}$ ), we find

$$\left. \begin{aligned} \frac{h}{2\pi} \int \frac{d\xi}{\xi} \sqrt{-n_3^2 + 4\gamma_1 \xi - \frac{1}{n^2} \xi^2 - EF\xi^3} &= n_1 h, \\ \frac{h}{2\pi} \int \frac{d\eta}{\eta} \sqrt{-n_3^2 + 4\gamma_2 \eta - \frac{1}{n^2} \eta^2 + EF\eta^3} &= n_2 h. \end{aligned} \right\} \quad (2)$$

When there is no electric field the integration can be easily performed and results in the relations  $\gamma_1 = (n_1 + \frac{1}{2}n_3)/n$ ,  $\gamma_2 = (n_2 + \frac{1}{2}n_3)/n$ ; hence  $n_1 + n_2$

\* *Physical Review*, 31, 973, 1928.



The numerical computation for the  $\eta$ -condition was made in the following way. We put :

$$\left. \begin{aligned} 12\gamma_2 n^4 EF &= 1 - k^2 ; \\ (-k^3 - \frac{3}{2}k^2 + \frac{1}{2}) &< \frac{27}{2}n_3^2 n^6 (EF)^2 \end{aligned} \right\} \quad (4)$$

is the condition for line radiation. The term on the right of the inequality is (the extreme case of very strong fields excepted) a small quantity ; hence the dissolution of the line takes place when the left-hand side  $\frac{1}{2} - \frac{3}{2}k^2 - k^3$  reaches a small positive value  $\Delta$ . For  $k$  increasing from 0 to 0.5 this function decreases from 0.5 to 0, whereas  $1 - k^2$  in the same time decreases from 1 to 0.75. The change therefore takes place for  $1 - k^2 = 0.75 + \frac{4}{9}\Delta$ , only a small amount above 0.75. In first approximation, hence, the dissolution of the line begins for

$$12\gamma_2 n^4 EF = 0.75( + \frac{4}{9}\Delta). \quad (5)$$

Introducing the numerical value  $12EF = 7.016 \times 10^{-7}F$ , and solving for  $\gamma_2$ , we find  $\gamma_2 = 106.9 \times 10^4 n^{-4}/F$ . So we have for

	$F = 535$	214	107	53	21	6.7 e.s.u.
for $n = 10$	$\gamma_2 = 0.20$	0.50	1.0			
$n = 20$		0.03	0.06	0.12	0.31	1.0
with	$\Delta = 0.05$	0.008	0.002 (for $n = 10$ ).*			

In this computation of  $\Delta$  we took  $n_3 = 2$  because transitions between the  $n$ th and the 2nd level are only possible for  $n_3 = 1, 2$  or 3. The quantity  $\gamma_2$  denotes a fraction of the total breadth of the Stark pattern of the line ; for  $n_3 = 1$  we have  $\gamma_2 = 1/2n$  for the extreme red components, increasing to  $1 - 1/2n$  for the extreme violet components. All the Stark components with  $\gamma_2$  below the values computed above from (5) persist as separate lines, whereas the components with  $\gamma_2$  above these values are dissolved. So we see that, with increasing field strength, first the extreme red components disappear, then gradually the dissolving process proceeds toward the smaller wave-lengths, and the extreme violet components are the last to be attacked. This is in accordance with the experiments of Rausch von Traubenberg,† who saw in a field of nearly  $F = 2300$  e.s.u. ( $= 700,000$  volt/cm.) the line  $H\gamma$  split up in a bundle of lines which in the inhomogeneous field was cut off obliquely, so that the violet components persisted longer than the red ones.

For the  $\xi$ -condition we introduce an analogous substitution :

$$12\gamma_1 n^4 EF = k^2 - 1 ; \quad k^3 - \frac{3}{2}k^2 + \frac{1}{2} > \frac{27}{2}n_3^2 n^6 (EF)^2. \quad (6)$$

For moderate fields, where  $k$  is a small amount above 1, we find that the transition point is independent of  $EF$  and for any field strength is determined by  $\gamma_1 = \frac{1}{2}n_3/n$ . For this value of  $\gamma_1$  the function  $f(\xi)$  with  $F = 0$  has a maximum exactly on the zero line, giving a constant  $\xi_1$  for this orbit. It is clear that

\* The condition for the limit of continuous radiation, given by R. and D. (form 21', p. 977), usually quoted in the simplified form:  $|W| < 2e\sqrt{eF}$ , corresponds to (5) for the case  $\gamma_2 = 1$ .

† *Die Naturwissenschaften*, 18, 417, 1930.

any trace of an electric field brings this maximum below the zero line and destroys the orbit, which is in fact the one giving rise to the extreme violet component of the Stark pattern. The other orbits can be dissolved by the  $\xi$  condition only in the case of strong fields, increasing the  $(EF)^2$  term above  $k^3$ . By elimination of  $EF$  we find

$$\gamma_1^2 > \left(\frac{n_3}{n}\right)^2 \frac{3}{32} \frac{(k^2 - 1)^2}{k^3 - \frac{3}{2}k^2 + \frac{1}{2}}.$$

Assuming, for instance,  $n_3/n = 0.1$  this leads to the numerical values

$k =$	2	5	10	20
$\gamma_1 >$	0.058	0.078	0.104	0.145
$F =$	7400	44000	140000	400000

These values of  $F$ , necessary to destroy only a small part of the pattern, are so large that they have no practical importance for our problem. So the  $\xi$ -condition does not affect the visibility of the lines in another way than already indicated for the extreme violet component.

2. It was not necessary in these computations to proceed to a second approximation, because for an exact treatment it is better to make use of the work of C. Lanczos.\* Although the subjects of the papers are apparently different—for Robertson and Dewey the origin of the continuous spectrum, for Lanczos the weakening of the lines—they treat, in reality, the same question, the transformation of line radiation into continuous radiation; and they proceed from the same equations. Their method of treatment, however, is different, one by quantum mechanics, the other by wave mechanics; the latter allows exact computations of intensities. In the wave-mechanical treatment there is no sharp limit between existence and non-existence of the line; before the quantum-mechanical limit is reached a kind of pre-dissolution begins to take place, and the existence of the line gradually dwindles after the limit is passed. Lanczos introduces a quantity  $\delta$ , the rate of dissolution of the orbit in the presence of an electric field, so that the probability of the  $n$ th state of energy is given by  $e^{-2\delta t}$ . This probability must be combined with that dependent on the spontaneous disappearance of the  $n$ th state, determined by  $e^{-2\alpha t}$ , where  $1/2\alpha = T$  is the mean lifetime of this state. The dissolution by the electric field is only perceptible if it takes place more rapidly than the normal disappearance, hence for  $\delta \gg \alpha$ . The number of atoms in this state,  $N$ , compared with the number in the case of absence of the field,  $N_0$ , is given by  $N = N_0\alpha/(\alpha + \delta)$ . Hence the intensity of the spectral line is

$$I = I_0/(1 + \delta/\alpha). \quad (7)$$

The rate of dissolution  $\delta$  depends on two integrals  $I_1$  and  $I_2$ , occurring in the Schrödinger function:

$$\delta = \frac{\pi e^2}{ha_0} \frac{e^{-2I_2}}{2I_1} = 2.066 \times 10^{16} \frac{e^{-2I_2}}{I_1}. \quad (8)$$

\* "Zur Intensitätsschwächung von Spektrallinien," *Z. f. Physik*, **68**, 204, 1931.

The exact computation of these integrals is a very complicated matter, since in strong electric fields the original quantum numbers themselves are distorted into "effective" quantum numbers, which must be found from series that are not convergent if the field strength is too large. It appears, however, that in such moderate fields as occur in ionized gases in celestial atmospheres these higher-order terms are rather insignificant, so that in the approximations used here these distortions of the quantum numbers may be neglected. Then we have to apply the following system of equations, given by Lanczos in his formulæ 51, 52, 53, 54, 58, 61, 64 (pp. 224-226 *loc. cit.*):

$$\left. \begin{aligned} \epsilon &= \frac{2a_0^2}{e}Fn^3; & \tau &= 8(n_2 + \frac{1}{2}(n_3 + 1))\epsilon = \sin^2 \phi, \\ k &= tg\frac{1}{2}\phi; & k' &= \sqrt{1 - k^2}, \\ 3\epsilon I_2 &= 2 \cos \frac{1}{2}\phi \cdot E' - 2 \sin \phi \sin \frac{1}{2}\phi \cdot K', \\ \epsilon I_1/n^2 &= 2 \cos \frac{1}{2}\phi(K - E), \end{aligned} \right\} \quad (9)$$

where  $E$  and  $K$  are the complete elliptic integrals of the first and second kind, and  $E'$  and  $K'$  are the same for the modulus  $k'$ . The factor occurring in  $\tau$  was written  $n_2 + \frac{1}{2}$  by Lanczos and restricted to the extreme components; the more general form  $n_2 + \frac{1}{2}(n_3 + 1)$  corresponds to what was called  $\gamma_2 n$  in the former treatment. We have here to consider that the treatment by wave mechanics gives for  $n_3$  the values 0, 1, 2 (for  $n=2$ ) instead of 1, 2, 3; this difference is compensated by the occurrence of  $n_3 + 1$  instead of  $n_3$  in (9).

Equation (7) shows that for  $\delta \gg \alpha$ ,  $I \ll I_0$ , hence the line has disappeared and is dissolved; for  $\delta \ll \alpha$ ,  $I = I_0$ , the line is unchanged. From equation (8) we see that for small  $I_2$  (the variations of  $I_2$ , because it is in the exponent, are the most important)  $\delta$  is large, of a much higher order than  $\alpha$  (which for low levels is about  $10^8$ ), hence the line is absent. With increasing  $I_2$ ,  $\delta$  decreases, and when its decreasing value passes  $\alpha$ , then the change from invisibility to normal intensity takes place rapidly: a rather small change in  $F$  suffices to bring it about. For the higher levels of hydrogen we may assume the value of  $10^6$  for  $\alpha$ ; the uncertainty as to its exact value has, as will be seen, no material influence. Moreover, it is not necessary to represent the rapid continuous change; we may replace it by a sudden disappearance at a certain value  $\log \delta = \log \alpha$  (which we will take equal to 6), the more so as the results will have to undergo some smoothing process afterwards. From the relations (8) and (9) the value  $\delta = 10^6$  determines a certain  $\tau$ , and so a certain value of  $n_2 + \frac{1}{2}(n_3 + 1) = \gamma_2 n$ . So, for a given field  $F$  and for a certain  $n$  the division mark between the visible and the dissolved part of the Stark pattern may be computed.

For this purpose corresponding values of  $z = 3\epsilon I_2$  and  $y = \log \epsilon I_1/n^2$  have been computed for different values of  $\tau$ ; they are collected in Table I.

There is a very smooth variation of  $z = 3\epsilon I_2$  with  $\tau$ ; for practical computations we may represent it by an interpolation formula

$$\tau = 1 - 0.60z + 0.05z^2. \quad (10)$$

TABLE I  
Values of  $3\epsilon I_2$  and  $\epsilon I_1/n^2$

$\tau$	$z=3\epsilon I_2$	$y=\log \epsilon I_1/n^2$	$\tau$	$z=3\epsilon I_2$	$y=\log \epsilon I_1/n^2$
$1-2.3 \times 10^{-8}$	$3.8 \times 10^{-8}$	+0.797	0.8830	0.197	-0.095
$1-1.4 \times 10^{-5}$	$2.3 \times 10^{-5}$	.601	0.850	.254	.147
0.99977	0.00041	.479	0.800	.340	.216
0.99880	0.00216	.386	0.750	.426	.279
0.99614	0.0066	.305	0.700	.515	.339
0.99038	0.0160	.230	0.600	.696	.455
0.9847	0.0255	.185	0.5868	.721	.472
0.9796	0.0342	.155	0.500	.882	.574
0.9698	0.0501	.110	0.4132	1.049	.686
0.940	0.101	+0.018	0.250	1.380	.951
0.920	0.135	-0.028	0.1170	1.678	1.313
0.900	0.170	.066	0.0301	1.902	-1.921

TABLE II  
Values of  $A$   
 $\log \alpha = 6$

$3.454\epsilon$	$n=9$	10	11	15	18	20	22	28	30	32
0.200	7.02	6.92	6.83	6.56	6.41	6.32	6.23	6.01	5.94	5.88
.100	6.75	6.65	6.56	6.27	6.10	6.00	5.91	5.69	5.62	5.56
.050	6.29	6.20	6.11	5.82	5.65	5.55	5.47	5.24	5.18	5.12
.040	6.15	6.05	5.97	5.68	5.51	5.42	5.33	5.11	5.04	4.98
.025	5.87	5.78	5.69	5.41	5.24	5.15	5.06	4.84	4.78	4.72
.020	5.74	5.65	5.56	5.28	5.11	5.01	4.93	4.71	4.64	4.58
.010	5.35	5.26	5.17	4.89	4.72	4.62	4.54	4.32	4.26	4.20
.005	4.97	4.88	4.80	4.51	4.35	4.25	4.16	3.94	3.88	3.82
.002	4.49	4.39	4.31	4.03	3.86	3.77	3.68	3.46	3.39	3.33

$\log \alpha = 8$										
	6.03	5.91	5.81	5.45	5.25	5.14	5.03	4.77	4.70	4.63
0.250	6.03	5.91	5.81	5.45	5.25	5.14	5.03	4.77	4.70	4.63
.200	5.60	5.49	5.40	5.09	4.90	4.80	4.70	4.46	4.39	4.33
.100	4.89	4.79	4.70	4.41	4.24	4.14	4.05	3.83	3.76	3.70
.050	4.38	4.28	4.20	3.91	3.74	3.65	3.56	3.34	3.28	3.32
.040	4.23	4.13	4.05	3.77	3.60	3.51	3.42	3.20	3.14	3.08
.025	3.94	3.85	3.76	3.48	3.32	3.22	3.13	2.92	2.85	2.80
.020	3.81	3.71	3.63	3.35	3.18	3.08	3.00	2.78	2.72	2.66
.010	3.41	3.31	3.23	2.95	2.78	2.69	2.60	2.39	2.32	2.26
.005	3.03	2.93	2.85	2.57	2.40	2.31	2.22	2.01	1.94	1.88
.002	2.54	2.44	2.36	2.08	1.92	1.82	1.74	1.52	1.46	1.40

For  $\tau = 0$ ,  $z = 2$  the real limiting gradient  $dz/d\tau$  is logarithmically infinite.

For  $\tau = 1$ ,  $z = 0$  the exact limiting gradient is  $-4\sqrt{2}/3\pi = -0.6002$ ; for our practical computations this discrepancy is irrelevant. It was not found possible to represent  $y$  by an interpolation formula. We have now to solve equation (8) for the point of disappearance,  $\delta = \alpha$ :

$$16.01 - 2 \times 0.4343 I_2 - \log I_1 = 6.0,$$

or

$$10.01 - y + \log \epsilon - \log n^2 = \frac{z}{3.454\epsilon}. \quad (11)$$

To find  $z$ , this equation, combined with the relation between  $y$  and  $z$  as given by Table I, was solved by graphical interpolation for different values of  $\epsilon$  and  $n$ . The results are given in Table II in the form of values of the quantity  $A = z/3.454\epsilon$ , as a function of  $n$  and  $3.454\epsilon$ . The computation has been repeated for  $\log \alpha = 8$ , to see the effect of different assumptions about  $\alpha$ .

Now that  $z = 3\epsilon I_2$  is known as a function of  $\epsilon$ , the interpolation formula (10) gives  $\tau$ ; from  $\tau$  and  $\epsilon$  the quantity  $\gamma_2 n$  is found by  $\tau = 8\gamma_2 n \epsilon$ . So we have

$$\gamma_2 n = \frac{1}{8\epsilon} (1 - 0.60z + 0.05z^2).$$

Inserting the values

$$\epsilon = 1.169 \times 10^{-7} F n^3 \quad \text{and} \quad z = 3.454\epsilon A = 4.040 \times 10^{-7} F n^3 A,$$

we find

$$\gamma_2 = \frac{106.9}{F} \left(\frac{10}{n}\right)^4 - 0.259 \frac{A}{n} + 0.0872 \times 10^{-5} \left(\frac{n}{10}\right)^2 A^2 F. \quad (12)$$

The first term is identical with the result of the former approximate calculation by means of equation (5).

The decrease of the number of visible higher members of a series with increasing electric field has been observed for some potassium series by Kuhn\* and by Bakker.† Miss Dewey measured the extension of the continuous spectrum to the red side of the series limit for helium.‡ Chalonge discussed the dissolution of the lower Balmer lines  $H\gamma - H\zeta$  into continuous emission in his researches on the continuous hydrogen spectrum.§ All these laboratory experiments with constant fields confirm the theoretical deductions qualitatively, though they do not allow an exact quantitative test of the formulas.

3. In a partly ionized gas we have to deal with variable electric fields, depending on the continually changing distances of the charged particles from the hydrogen atoms. The probability of a field  $F$  is given by Holtsmark's function  $\bar{W}(\beta)$ , where  $\beta = F/F_0$ , and where  $F_0$ , a certain

\* *Zs. f. Physik*, **61**, 805, 1930.

‡ *Phys. Review*, **35**, 155, 1935.

† *Proc. Amsterdam*, **36**, 589, 1933.

§ *Ann. de Phys.*, **XI**, 1, 123, 1934.

mean or normal field strength, depends on the physical parameters of the gas :  $F_0 = 2.61N^{2/3}e$  ( $N$  = number of charged particles per unit volume)  $= 46.8(2P/T)^{2/3}$  ( $P$  = electron pressure). The function  $\bar{W}(\beta)$  has been computed by Verwey.\* For a certain  $F$  or  $\beta$  the pattern of Stark components is extended over a breadth given by

$$\Delta\nu/\nu = \Delta\lambda/\lambda = 3EF\{n(n-1) + 2\} \left( \frac{1}{4} - \frac{1}{n^2} \right)^{-1},$$

or

$$\Delta\lambda = \pm 0.00256 \left( \frac{n^2}{n^2 - 4} \right)^2 \{n(n-1) + 2\} F = \pm SF. \quad (13)$$

For the lower members of the Balmer series, where the intensity of each Stark component is known, the resulting distribution of intensity over the broadened line could be exactly computed. For the higher members these intensities are not known. Hence we will assume that for each line the total intensity is evenly distributed over a band with a breadth  $2B = 2\beta B_0$ , where  $B_0 = SF_0$ . To each value of  $\beta$  (*i.e.* between  $\beta$  and  $\beta + d\beta$ ) belongs the corresponding breadth  $2\beta B_0$ , and its intensity at each point is given by  $\bar{W}(\beta)d\beta : 2\beta B_0$ . If there was no dissolution of the Stark components the total intensity for a certain  $\Delta\lambda$  would be found by integrating over all values of  $\beta$ , from the lowest  $\beta_1 = \Delta\lambda/B_0$ , for which the extremity of the band just reaches this  $\Delta\lambda$ , up to  $\infty$ .

$$I = U(\beta_1) = \int_{\beta_1}^{\infty} \frac{\bar{W}(\beta)}{\beta} d\beta; \quad \int_{-\infty}^{+\infty} U(\beta_1) d\beta_1 = 1. \quad (14)$$

Now a certain part of the red side of each band is dissolved and changed into continuous radiation ; this means that part of each band is cut off at a point given by the fraction  $\gamma_2$  of the total breadth  $2\beta B_0$ . We may represent these conditions by a diagram (fig. 2), where the abscissa represents  $\Delta\lambda$ , and  $\beta$  is counted from 0 upward, and the breadth of the Stark band for every  $\beta$ , proportional to  $\beta$ , is included between the two inclined straight limiting lines  $r$  and  $v$ . Each horizontal line between these limiting lines represents the band for  $\beta$ , its vertical co-ordinate, and must be provided with an intensity value  $\bar{W}(\beta)/\beta$ . We get the total intensity  $U(\beta_1)$ , as given by (14), for some  $\Delta\lambda$  by integrating along a vertical line, corresponding to this  $\Delta\lambda$ , the intensity values upward from the point, where the vertical intersects the limiting line, *i.e.* from  $\beta_1$ . Without the dissolution of line radiation the integration would extend to  $\infty$  ; now, however, the red side part of the whole diagram is cut off by a clipping line, intersecting each horizontal band at a distance  $\gamma_2 \times 2B = 2\gamma_2\beta B_0$  from its left-hand end, where

$$\gamma_2 = \frac{107 \left( \frac{10}{n} \right)^4}{\beta F_0} - 0.259 \frac{A}{n} + 0.0872 \cdot 10^5 \left( \frac{n}{10} \right)^2 A^2 \beta F_0,$$

or

$$2\gamma_2 B = 2B_0 \left\{ \frac{107 \left( \frac{10}{n} \right)^4}{F_0} - 0.259 \frac{A}{n} \beta + 0.0872 \cdot 10^5 \left( \frac{n}{10} \right)^2 A^2 \beta^2 F_0 \right\}. \quad (15)$$

\* *Public. Amsterdam*, 5, Table 3, p. 17.

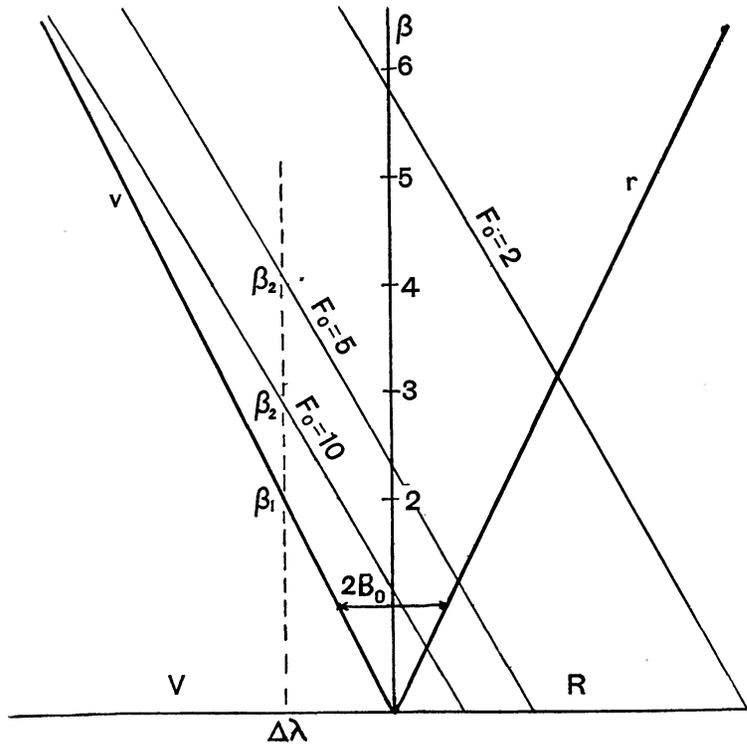


FIG. 2.

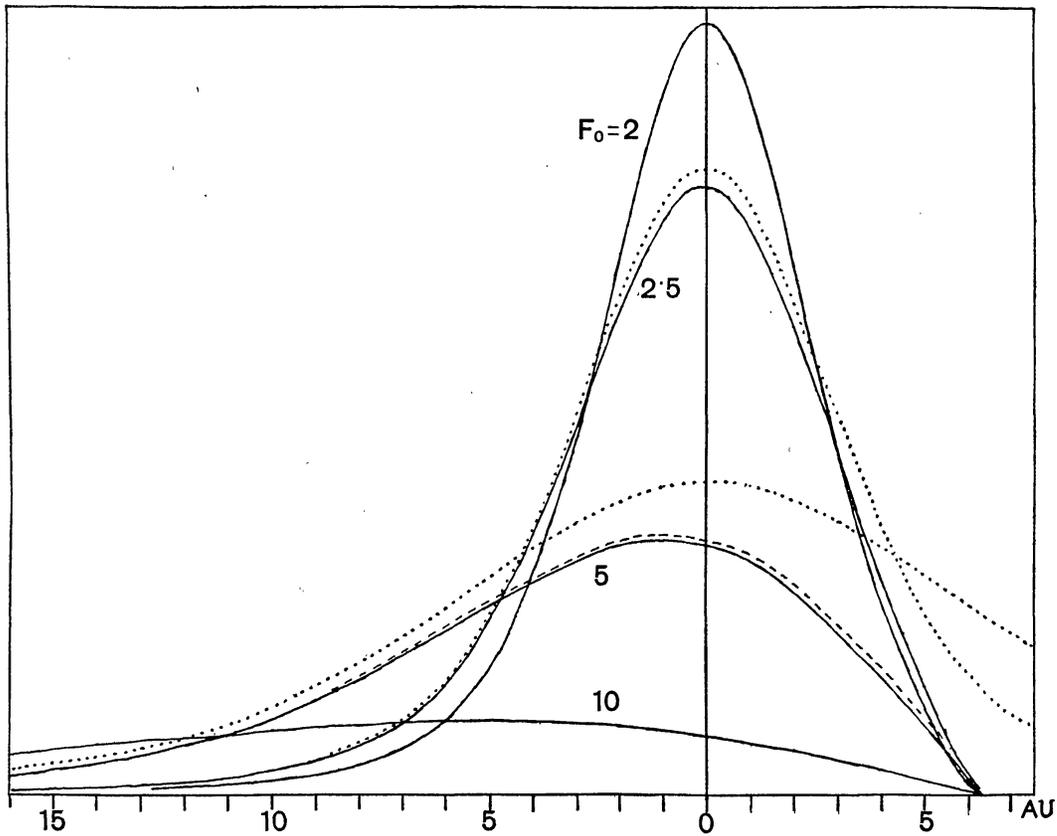


FIG. 3.

For a given value of  $F_0$  (and  $n$ ) this function of  $\beta$  can be computed; the values of  $A$  are taken from Table II, representing  $A$  as a function of  $\log \epsilon = \log 1.17 \times 10^{-7} n^3 F$ . Then the clipping lines are drawn in the diagram (in fig. 2, standing for  $n=20$  they are inserted for  $F_0=2, 5$  and  $10$ ), running as nearly straight lines from the red side below to the violet side upward, so that the part dissolved at the red side is at the same time situated at the upper side. For each  $\Delta\lambda$  the integration now extends from  $\beta_1$ , not to  $\infty$  but only to the intersection point  $\beta_2$  with the clipping line. The fraction  $\int_{\beta_1}^{\beta_2} : \int_{\beta_1}^{\infty}$  denotes the part of the total brightness left at this point  $\Delta\lambda$  of the broadened line.

TABLE III  
*Intensity Distribution in Balmer Lines*

$F_0$	8	7	6	5	4	3	2	1	0	1	2	3	4	5	6
	$n=10$														
200						1	40	139	118						
100				2	8	29	110	286	326						
"			(1	5	12	35	120	304	352)						
70			2	6	14	41	135	344	450	149					
50		1	3	8	19	49	150	376	522	302					
"		(1	3	9	20	51	153	380	533	322)					
35	2	3	5	10	22	54	158	391	555	373	113				
25	3	4	6	12	23	57	162	397	565	393	152	37			
20	3	5	7	13	25	58	164	399	568	397	160	51	13		
10	4	5	8	14	26	60	166	402	572	402	166	59	25	13	7
	$n=20$														
10			1	3	9	28	100	239	212						
7		1	2	6	13	38	124	307	341						
5	1	2	4	8	17	46	141	352	455	127					
"	(1	2	4	8	18	47	144	358	465	144)					
3.5	2	3	5	10	20	52	153	379	527	301					
2.5	3	4	6	11	22	55	159	392	556	371	100				
"	(3	4	6	11	23	56	160	393	558	376	109)				
2	3	4	7	12	23	56	162	396	563	388	141	12			
1.4	4	5	7	13	24	58	164	399	568	399	164	51	12		
1	4	5	8	14	26	59	165	401	571	400	164	57	22	9	2
	$n=30$														
2.5			1	4	9	27	93	214	162						
2		1	2	5	12	33	108	256	228						
"		(1	2	5	12	35	114	270	247)						
1.4		1	3	7	15	40	129	317	356						
1	2	2	4	9	18	48	145	358	466	136					
"	(2	2	4	10	19	49	147	362	477	150)					
0.5	2	3	5	11	23	56	160	393	558	374	109				
0.35	3	4	7	13	24	58	163	398	567	394	154	38			
0.25	4	5	8	14	26	60	165	401	571	401	164	57	22	10	2

The integrations have been performed, for a number of different values of  $F_0$  for each of the quantum numbers  $n$  of Table II, for every tenth part of  $2B_0$ . An extract of the results for the intensity, for entire multiples of  $B_0$ , is given in Table III for  $n = 10, 20, 30$ . (The lines in parentheses hold for  $\log \alpha = 8$ .) Each line represents an intensity curve for that  $n$  and  $F_0$ . The scale of the abscissæ must be transformed into a scale of wave-lengths by means of  $B_0 = SF_0 AU$ . As an example, the intensity curves for  $n = 20$  are represented in fig. 3. The dashed line for  $F_0 = 5$  holds for  $\log \alpha = 8$ ; it shows that errors in the assumed value of  $\alpha$  have only a small effect. A dotted line for  $F_0 = 5$  and for 2.5 indicates how the original curve without dissolution of line radiation would have run. By this asymmetrical dissolution the maximum and the barycentre of each line are displaced towards the violet side.

The intensity curves for different  $F_0$  are cut off at the red side all at nearly the same distance in  $AU$ . This limit is determined in fig. 2 by the intersection point of the clipping line with the right-hand side limiting line. If we assume the former to be exactly straight, corresponding to the first two terms of (15) with constant  $A$ , then the intersection point is given by  $(107/F_0)(10/n)^4 : 2(1 + 0.259A/n)$  in units of  $B_0$ ; if the scale is transformed into  $AU$  by substituting  $B_0 = SF_0$ , we see that  $F_0$  disappears from the result. We find, by means of these expressions, for the red side limit of the curves :

for $n =$	9	10	11	15	18	20	22	28	30	32
	28.9	23.6	19.7	10.9	7.7	6.3	5.2	3.27	2.86	2.52 $AU$ ,

roughly varying with  $1/n^2$ .

By integrating the intensity in the tables and curves we find what fraction of the total intensity of the line remains as line radiation, while the other part is dissolved into continuous radiation. The directly computed results are compiled in Table IV.

TABLE IV  
*Theoretical Intensities of Balmer Lines in Percentages of Normal Intensities*

$n$	$F_0 = 200$	100	70	50	35	25	20	10	7	5	3.5	2.5	2
9	26	59	75	86	93	96	98	99		100			
10	15	40*	57	74*	86	92	95	99		100			
11	8	26	40	56	73	85	90	97		98			
15				14	23	35	45	76	87	93	97	98	99
	$F_0 = 10$	7	5	3.5	2.5	2	1.4	1	0.7	0.5	0.35	0.25	
18	45	61	76	87	93	96	97	99	99	100			
20	29	43	58†	74	85†	90	95	97	99	100			
22	19	29	42	57	72	81	89	94	97	98			
28					33	41	57	72	84	92	96	98	
30					25	32‡	45	60‡	75	86	93	96	
32					18	24	36	49	64	78	88	94	

\* 44 and 76.

† 60 and 86.

‡ 34 and 62 for  $\log \alpha = 8$ .

This percentage of residual intensity can be represented rather exactly as a function of  $\log F_0 + 3.8 \log (n/20)$ ; for values of this logarithm written in the next line the percentage takes the value of the second line:

$\log F_0 \left(\frac{n}{20}\right)^{3.8}$	-0.5	-0.2	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.5
percentage	100	99	97	96	94	91	85	78	68	58	48	37	27	13	6

By means of this little table we can find for every condition of the gas, indicated by  $F_0$ , how the residual intensities of the Balmer lines decrease with their increasing quantum number, till they disappear.

4. There is, however, another circumstance to be considered here, the merging of the consecutive members of the Balmer series. If the electric field increases in strength and the Stark pattern broadens, the patterns of adjacent lines at last will eventually overlap. The breadth of the Stark pattern is given by  $h\Delta\nu = 3ea_0F\{n(n-1) + 2\}$ ; the distance of two adjacent lines for large  $n$  is given by  $h\Delta\nu = e^2/a_0n^3$ . They are equal for

$$3a_0^2eF\{n^4(n-1) + 2n^3\} = 1 \quad \text{or} \quad F = (1/3E)\{n^4(n-1) + 2n^3\}^{-1};$$

for  $n = 10, 20, 30$  this formula gives  $F = 62, 1.9, 0.24$ . In this case of overlapping it is not permissible in physical theory to treat the Stark effect of each line separately; the collectivity of lines must be treated together. In our case the result of this treatment must be smoothed afterwards into a continuous intensity curve produced by continually changing fields. We may assume that the result will be the same, if first we treat each line separately, compute the smoothed result of changing fields, and then combine the results for the different lines.

It is readily seen that lines appreciably dissolved, completely merge into a continuous spectrum of nearly constant intensity; it is only for much weaker fields where the dissolution is small that the lines begin to be separately visible. The relation between these two causes of disappearance of the lines depends on the principal quantum number; the merging of adjacent lines depends on  $Fn^5$ , the dissolution on  $Fn^4$ , hence for the lower members the dissolving process will be more important than for the higher members of the series.

A combined intensity curve of all the lines was computed and constructed for  $F_0 = 20$  in the vicinity of  $n = 20$ , for  $F_0 = 2$  and  $1.4$  in the vicinity of  $n = 20$ , and for  $F_0 = 0.2$  in the vicinity of  $n = 30$ . For the values of  $n$  contained in Table II the intensity curves were taken from direct computation; for the other values they were found by interpolation, using  $F_0n^4$  as the argument. The normal total intensity for each was assumed to be  $\sim 1/n$ , because in conditions of equilibrium the population of the higher levels increases with  $n^2$ , and the transition probabilities approach  $n^{-3}$ . The result is shown in figs. 4-6. The separate curves of the spectral lines are given by dots (in fig. 5 they are only given for  $F_0 = 2$ ), the combined curve is given in full (for  $F_0 = 1.4$  in fig. 5 by a dashed line). We see that with decreasing wave-length the undulations rapidly decrease and disappear, and a continuous radiation remains. Connecting the constant part with

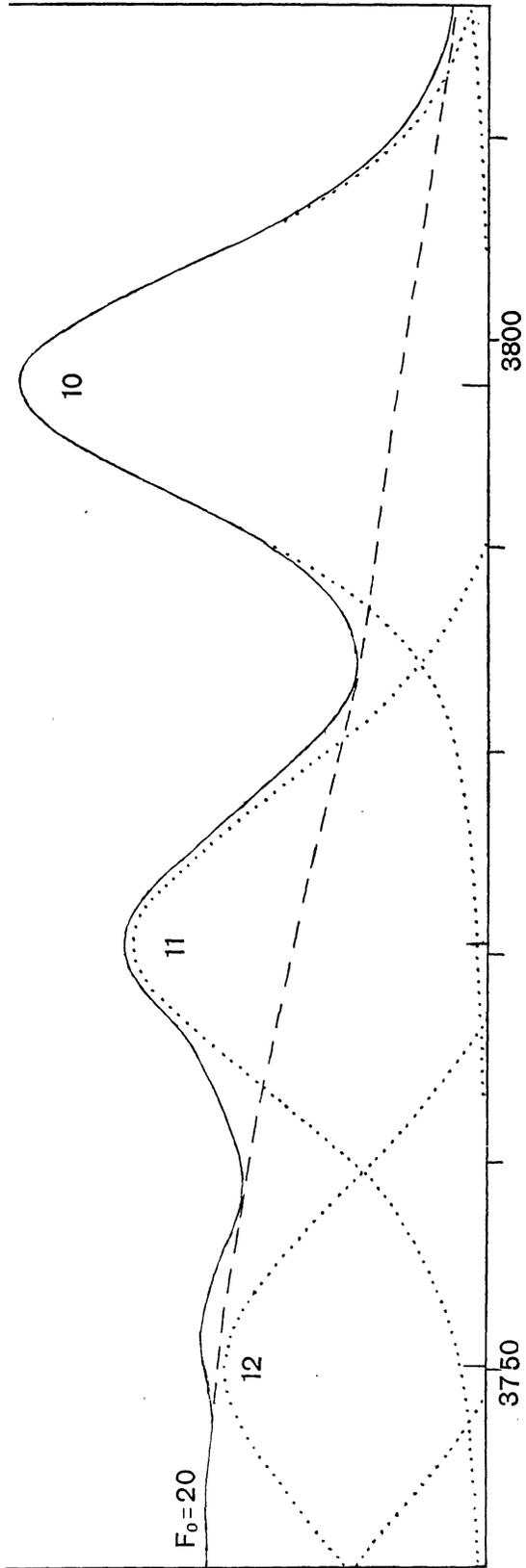
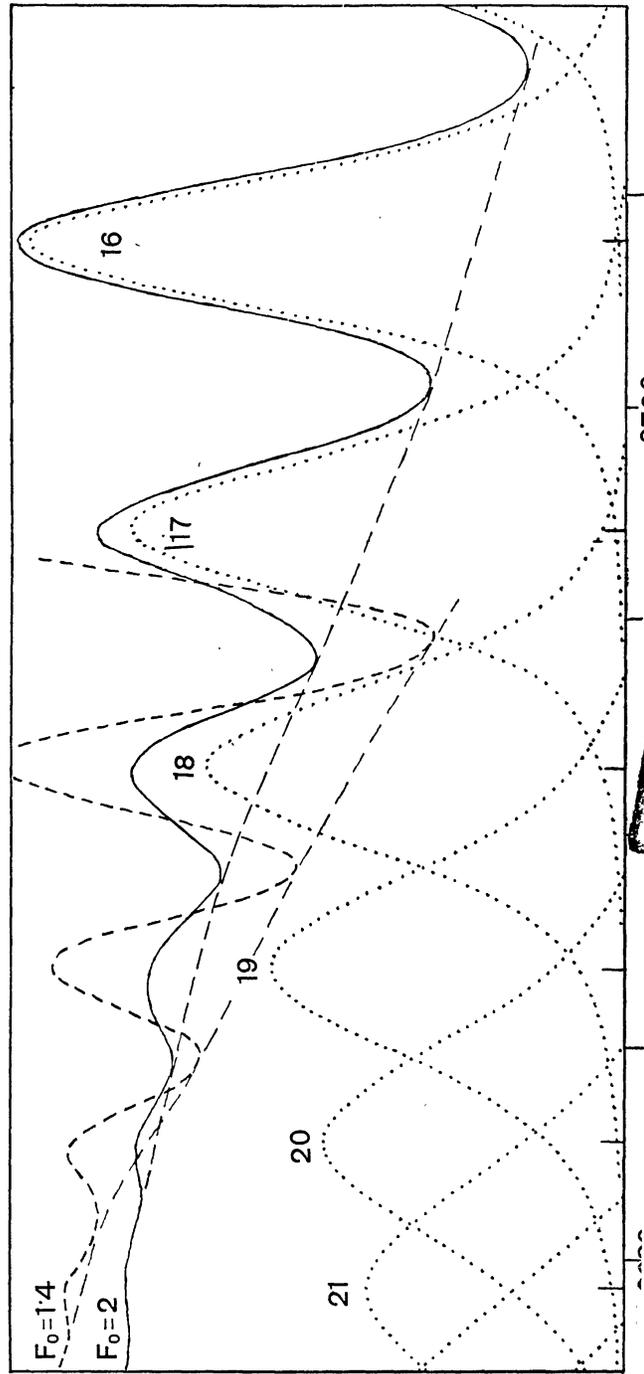


FIG. 4.



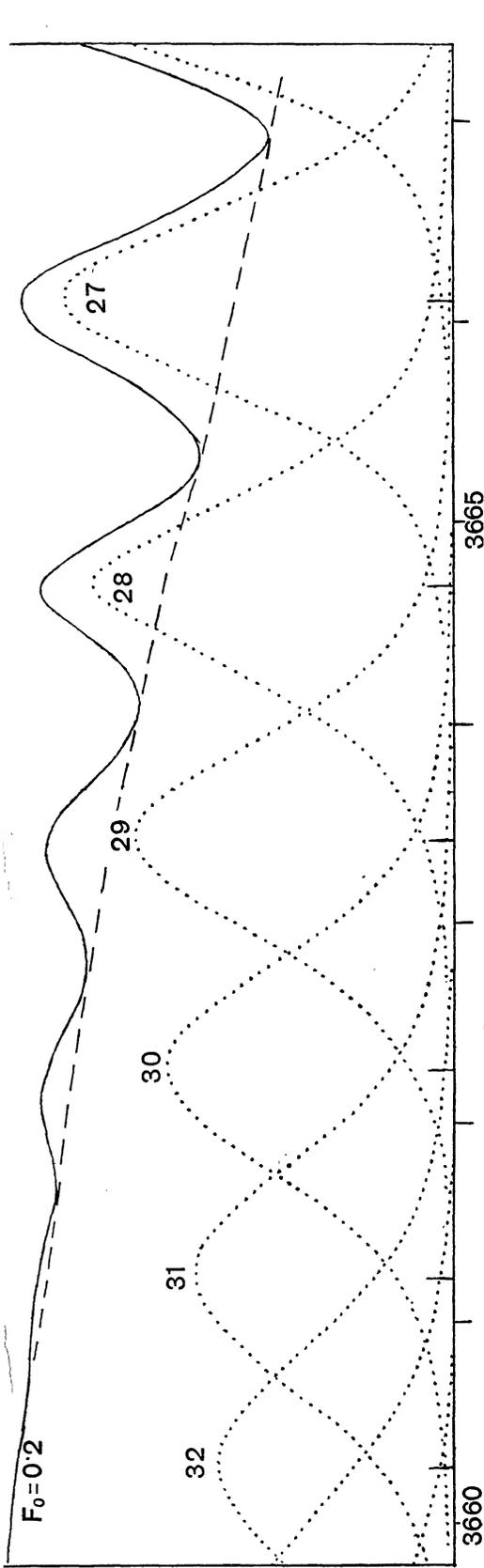


FIG. 6.

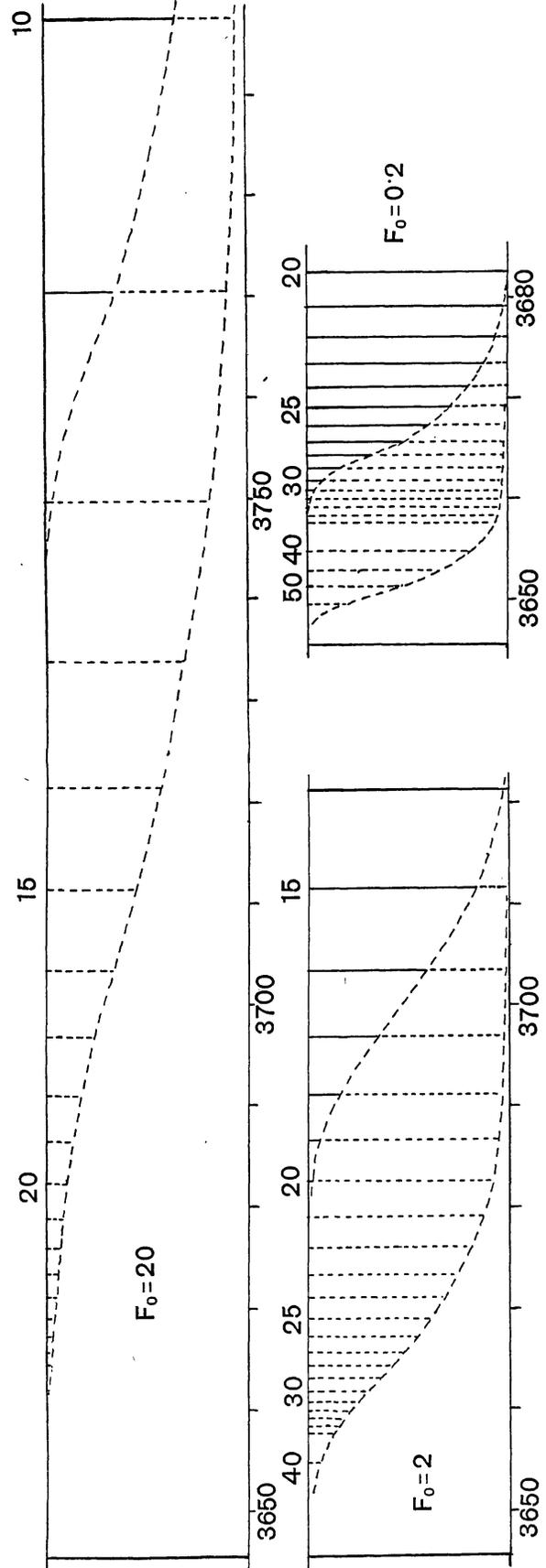


FIG. 7.

# ON A GENERALIZATION OF LINDBLAD'S THEORY OF STAR-STREAMING.

*S. Chandrasekhar.*

1. *Introduction.*—The dynamics of a stellar system can be studied by one of two methods. The first method consists in specifying in some detail the character of the orbits described by the individual stars and then seeking for a distribution function  $f$  of the space and the momenta coordinates which are consistent with the derived motions and the equation of continuity. In the second method, on the other hand, no attempt is made to describe the trajectories of the individual stars, and the equations of motion are discussed only to obtain the first integrals consistent with the general large scale symmetry character (*e.g.* spherical, cylindrical, etc.) of the system; the distribution function then follows from an appeal to Jeans's theorem. As an example of an application of these two methods we may refer to the manner in which the dynamics of the globular clusters has been studied. In Eddington's\* investigation the first method was adopted. However, as Shiveshwarkar † has shown, results identical with Eddington's can be obtained by the second method. In this particular instance Shiveshwarkar's solution of the dynamics of the globular clusters is mathematically the more elegant. But Eddington's treatment gives greater insight into the actual state of motions in the globular cluster—in particular, it makes the dependence of the ratio of the axes of the velocity ellipsoid on the distance ‡ easily understood. When we come to treat the dynamics of the galaxy the situation is not so simple. As a model of the galaxy we may consider a stellar system with cylindrical symmetry. But an application of Jeans's theorem shows that for such a system radial star-streaming is not possible. This is in direct disagreement with the results of observation. At the same time, as Oort § and also Shiveshwarkar || have shown, the dynamical properties of stellar systems with cylindrical symmetry have certain striking resemblances with the state of motions in the galaxy. In particular, it can be shown ¶ that

$$\frac{|\Theta - \Theta_0|}{|\bar{\Pi}|} = \sqrt{1 + \frac{A}{\omega_0}}, \quad (1)$$

where  $\Theta_0$  and  $\omega_0$  are the linear rotational and angular velocities respectively, and  $\bar{\Pi}$  the mean speed in the radial  $\varpi$  direction and  $A$  the Oort

\* A. S. Eddington, *M.N.*, **74**, 5, 1913; **75**, 366, 1914.

† S. W. Shiveshwarkar, *M.N.*, **96**, 749, 1936.

‡ The relation in question is:  $\bar{R} : \bar{\Theta} : \bar{\Phi} = (1 + kr^2)^{\frac{1}{2}} : 1 : 1$ .

§ J. H. Oort, *B.A.N.*, **4**, 269, 1928.

|| S. W. Shiveshwarkar, *loc. cit.*

¶ Shiveshwarkar, *loc. cit.*, p. 757.

used, as given *loc. cit.*, Table I, pp. 90–91. Their  $\log E$  was compared with what they should be after simple theory. Since the stimulated transitions up and down between the highest levels, the ionized state included, are more important than the spontaneous transitions (cf. *M.N.*, 96, 788), the partition between these levels will probably be nearer to the thermodynamical equilibrium than to the case of a pure capture spectrum. Hence in this first approximation a normal decrease  $\sim n^{-1}$  was assumed. In this way the line number was derived where the residual intensity was 50 per cent. For the three columns of data we found  $n=30$  for film 49a,  $n=26$  for 48a,  $n=28$  for 49b. From the values of Table V this corresponds to  $F_0=0.10, 0.20, 0.14$ . Assuming  $F_0=0.15$ , we find from the expression  $F_0=2.61N^{2/3}e$  that  $\log N=12.10$ . This is the total number of charged particles, ions and electrons together per c.c. Taking their numbers equal, we have  $\log N_e=\log N_i=11.80$ . Assuming a temperature of  $5000^\circ$ , we find the combined pressure  $2P=0.9$  dynes/cm.<sup>2</sup>.

From the intensity of the Balmer continuum, making use of the theoretical value of the absorption coefficient, Menzel and Cillié found  $\log N_i N_e=23.18$  (*Harv. Circ.* No. 410, p. 23), hence, in the case of a pure hydrogen chromosphere  $\log N_i=\log N_e=11.59$  (for  $T=10,000^\circ$ ; for  $6000^\circ$ ,  $\log N=11.42$ ). The results are accordant as to the order of magnitude. An exact correspondence cannot be expected, first because Menzel and Cillié's result holds for the base of the chromosphere, whereas our value represents average conditions. Moreover, in the emission of the Balmer continuum only the interaction of hydrogen ions with the electrons plays a rôle, whereas the Stark broadening depends on all the charged particles, which are probably produced in large measure by the ionization of metal atoms.

The problem of deriving the behaviour of the Balmer series as absorption lines in ordinary stellar spectra, by combining our results for the different layers of an atmosphere, is left for further researches.

*Summary.*—The theoretical work of Robertson and Dewey, and of Lanczos on the dissolution of the highest levels of an atom by electric fields is applied to the disappearance of the higher members of the Balmer series. Their intensity curves in the emission spectrum of a partly ionized gas are computed. The merging of the highest Balmer lines into continuous radiation allows the derivation of the electron density in an atmosphere by means of the number of visible lines. This is applied to the solar chromosphere.

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