Professor H. S. Mendenhall, Oklahoma Agricultural and Mechanical College, Stillwater, Okla.
Mr. Willard F. Mullen, Pennsylvania State College, State College, Pa.
Mr. Ali M. Naqvi, Harvard College Observatory, Cambridge 38, Massachusetts.
Dr. Guy C. Omer, Jr., California Institute of Technology, Pasadena 4, Calif.
Dr. A. Keith Pierce, University Observatory, Ann Arbor, Michigan.
Dr. Stefan Piotrowski, Harvard College Observatory, Cambridge 38, Mass.
Mr. Grote Reber, Box 4868, Cleveland Park Station, Washington, D. C.
Miss Nancy G. Roman, Yerkes Observatory, Williams Bay, Wisconsin.
Mr. Winfield W. Salisbury, Collins Radio Co., Cedar Rapids, Iowa.
Professor Charles L. Seeger, Franklin Hall, Cornell University, Ithaca, N. Y.
Dr. H. K. Sen, Harvard College Observatory, Cambridge 38, Mass.
Mr. Richard E. Strikler, 3415 38th St. N. W., Washington 16, D. C.
Mrs. Constance S. Warwick, Harvard College Observatory, Cambridge 38, Mass.
Mr. James W. Warwick, Harvard College Observatory, Cambridge 38, Mass.
Dr. Albert G. Wilson, 78 Sierra Bonita Ave., Pasadena 4, California.
Mr. Stanley P. Wyatt, Jr., Harvard College Observatory, Cambridge 38, Mass.

The Planetary Theory of Kepler

By A. PANNEKOEK

I

Kepler's planetary theory is set forth in his "Astronomia Nova" with subtitle "de Motibus Stellae Martis" (On the motions of the star of Mars). This work, made accessible to modern astronomers in 1929 through a careful German translation by Max Caspar, shows a special character different from most of the great works of science. Usually in the publication of new important researches only the results with the data and arguments are given; the discoverers keep to themselves how they arrived at them, their fruitless endeavors, their detours, their failures, and exhibit the result as a well-rounded harmonious structure, as a work sometimes of art, constructed straight forward, where all traces of the difficult searching have been effaced. Thus Copernicus, Newton, Laplace, Gauss. This is fine for study and admiration. But in this way outsiders get a wrong idea of the making of science; they do not suspect, what every scientific worker knows through his own practice, how many painful failures and long detours one must go through before finally the direct way is found which then afterwards is easily seen as the obvious truth. Kepler, differently, exposes his entire course of research, his errors, his false suppositions and their disclosure, his perplexities and new endeavors, till the simple truth springs forward; all is laid open before the reader. Thus the book gives a true image of the growth of scientific discovery; here we see, as it were, the scientific worker in overalls.

Kepler himself in the introductory summary says of his method: "When Christopher Columbus, Magelhaens or the Portuguese, of whom the first discovered America, the second the Chinese sea, the third the way around Africa, tell us of their wanderings, we not only forgive them but would not wish to miss their stories, because then we should miss the great pleasure contained therein. So it will not be judged a fault in me that I do likewise out of the same regard for the reader. It is true that in reading of the difficulties of the Argonauts we do not experience them ourselves, whereas the difficulties and thorns in my researches will affect the reading itself. But that is the common lot of all books on mathematics, and since among men some are pleased with this, others with that, there will be some who will feel a great joy when after having vanquished the difficulties of understanding they have the entire series of my discoveries before their eyes." In reading his book we are indeed partaking in an adventurous voyage of discovery that opened a new world of astronomical progress.

Some introductory parts, the dedication to the emperor Rodolphe II, a panegyric on Tycho Brahe, a paragraph "to the reader" by Tengnagel, Tycho's son-in-law, remind us of the often difficult conditions under which the work came into being. We remember how Kepler through his first publication "Mysterium Cosmographicum"—wherein the distances of the planets from the sun were related to the five regular solids—came into first contact with Tycho; but only after Tycho had settled at Prague and Kepler by religious persecution was obliged to give up his teaching job at Graz in Steyermark, could he accept Tycho's invitation to join him and to assist in the reduction of his observations. Through Tycho's influence he was appointed "Imperial Mathematician"; and when in the next year (1601) Tycho died, he was ordered to continue Tycho's work with the instruments, under which he judged the results of observation and their reduction to be included. This brought him into conflict with Tycho's heirs who suspected that he, as a Copernican, sceptical against Tycho's world system, would not have sufficient regard for the honor of Tycho. An agreement was reached that Kepler should discuss the observations of Mars, and Tengnagel should prepare the final planetary tables—for which, however, the latter, as a high state official, lacked the time as well as the capacity. When in 1605 Kepler was ready with his work, Tengnagel after much delaying at last had to consent to the publication. Lack of money in the imperial treasury was a cause of still more delay, so that it could not appear until 1609. In his dedication, where in allegorical language Kepler tells the emperor that he brings him the war-god as a well-fettered captive, he reminds him that the campaign has to be pursued against Mars' relatives, the other planets, so that more money is needed. The emperor had the disposal of the printed issue; since, however, Kepler had not received salary for many years and lived in great distress, he took the right himself to dispose of the printed copies, and to publish and send them to the scholars of his acquaintance.

II

Kepler starts with explaining that the apparent motion of a planet presents two irregularities, one depending on its own longitude in the ecliptic, the other, producing the alternation of direct and retrograde motion, depending on its place relative to the sun. To determine the former by eliminating the latter, oppositions are always used. But opposition to what? Ptolemy, and in his wake Copernicus and Tycho Brahe, took the opposition to the mean sun. It means that as point of reference is taken the centre of the sun's orbit with Ptolemy, of the earth's orbit with Copernicus. Kepler, however, in his first work had already emphasized that always the real sun has to be used; because the sun is not only the source of all light but also the source of power that moves the planets.

Does it make any difference? If the planet describes its orbit with constant velocity, with the sun as well as the centre of the earth's orbit different from the centre of the planet's orbit, we can make use of either of these two points in the same way. Distance of the planet's orbital centre and direction of the line of apsides differ according as to which of them is chosen as zero point of reference; but both will fit equally well. Now, however, the velocity in the planet's orbit is not constant. Ptolemy's *punctum aequans* means that in perihelion the planet moves faster, in aphelion more slowly. Now the line of apsides is a real line of symmetry in the fluctuation of the planet's velocity; so observations must be able to decide whether the line of apsides passes through the real sun or through some other point, outside the sun. The same holds for the line of nodes, the line of intersection of the plane of the orbit with the ecliptic; it must be possible to derive from the observed latitudes whether the line of nodes passes or not through the sun.

Then he speaks of the second irregularity, depending on the sun. "Thereover people have much astonished themselves, and each has advanced another explanation . . . Latin authors supposed that in the aspects and rays of the sun a force lived attracting really the other planets . . . Ptolemy made his explanation an object of numerical and geometrical treatment, but the astronishment was not removed thereby. . . . Copernicus, however, with the ancient Pythagoreans and Aristarchus, and with them myself also, we contest that the second inequality belongs to the proper motion of the planet, and we hold it to be only an appearance due to the yearly revolution of the earth about the immobile sun." And Tycho's system is strikingly characterized thus: "Ptolemy put epicycles upon the excenters, and Brahe, on the contrary, put all excenters upon one single epicycle that is the orbit of the sun."

And, he continues, "I will in the following demonstrations apply all three forms from these authors. Tycho, when I advised him to do so, replied that he would have done so by himself, also if I had been silent. And when dying he asked me, whom he knew to adhere to the Copernican system, to accomplish all the demonstrations for his system also." This Kepler did. For us the reading is thus made more cumbersome, but for his time, when among astronomers there was much strife over the true world-system, it was certainly necessary.

III

In the second part of his book Kepler tells first how it came about that he should treat the motion of Mars. In all these apparently chance events he sees a Divine Providence, leading him to Prague, where Tycho's pupil Longomontanus was occupying himself with Mars, but could not succeed and then instead took up the moon's motion. Kepler had some misgivings about erroneous methods of reduction and began by testing carefully all corrections. From Tycho's observations he found the parallax of Mars never to exceed 3′, though up to that time the much smaller solar parallax had been assumed always to be 3′. He found by different methods that the inclination of the orbit was always the same, nearly 1° 50′, hence all assumptions of a variable inclination, which would have complicated everything, could be rejected. He derived anew all the oppositions from 1580 to 1600, now oppositions to the sun itself, and he completed the list with the oppositions of 1602 and 1604 deduced from observations by himself and his friend David Fabricius at Emden. This table of oppositions formed the basis further on of all his work. They afford the longitudes of Mars at different moments as seen from the sun. The variations in angular velocity are visible here at first sight.

Table of Oppositions of Mars

Time				Longitude	Latitude	Long. Computed	Difference	Lat. Comp.
			h m	° ′ ″	° ′	° ′ ″	′ ″	° ′
1580	Nov.	18	1 31	66 28 35	+1 40	66 28 44	—0 9	+1 $45\frac{1}{2}$
1582	Dec.	28	3 58	106 55 30	+4 6	106 57 4	—1 34	+4 $3\frac{1}{3}$
1585	Jan.	30	19 14	141 36 10	+4 $32\frac{1}{6}$	141 37 46	—1 36	+4 $30\frac{1}{2}$
1587	March	6	7 23	175 43 0	+3 41	175 43 16	—0 16	+3 37
1589	Apr.	14	6 23	214 24 0	+1 $12\frac{3}{4}$	214 26 12	—2 12	+1 $5\frac{1}{3}$
1591	June	8	7 43	266 43 0	—4 0	266 43 51	—0 51	—3 $59\frac{1}{6}$
1593	Aug.	25	17 27	342 16 0	—6 2	342 16 42	—0 42	—6 $3\frac{3}{4}$
1595	Oct.	31	0 39	47 31 40	+0 8	47 31 54	—0 14	+0 $5\frac{1}{5}$
1597	Dec.	13	15 44	92 28 0	+3 33	92 28 3	—0 3	+3 20
1600	Jan.	18	14 2	128 38 0	+4 $30\frac{5}{8}$	128 38 18	—0 18	+4 $30\frac{1}{4}$
1602	Feb.	20	14 13	162 27 0	+4 10	162 25 13	+1 47	+4 $7\frac{2}{5}$
1604	March	28	16 23	198 37 10	+2 26	198 36 43	+0 27	+2 $18\frac{3}{5}$

Ptolemy had represented this variation by a *punctum aequans* situated at a distance $2e$ from the sun (in his case the earth), with the centre of the circle exactly midway between them. Though Kepler in his "Mysterium Cosmographicum" had given reasons why this bisection should be exactly correct, he wished to test it by means of Tycho's highly accurate places. Ptolemy had needed three oppositions; since Kepler had to determine one unknown more, the ratio of division of the distance sun-equant by the circle's centre, he had to use four data. For four chosen moments (the oppositions of 1587, 1591, 1593, 1595) he knew the directions as seen from the sun, as well as the directions as seen from the equant since the latter increase proportionally to the elapsed time. The problem to find from these data the direction of the line of apsides and the two distances, from the circle's centre to the sun and to the equant, cannot be solved by a direct method; Kepler had to solve it by trying various suppositions, in successive approximations. "If this cumbersome mode of working displeases you" he says to the reader "you may rightly pity me, who had to apply it at least 70 times with great loss of time; so you will not wonder that the fifth year is passing already since I began with Mars. . . Acute geometers equal to Vieta may show that my method is not at the level of art. . . May they solve the problem geometrically. For me it suffices that . . . to find the way out of this labyrinth, instead of the torch of geometry I had an artless thread guiding me to the exit." The result of these computations was that the total eccentricity amounts to 0.18564, that the sun is 0.11332 and the equant 0.07232 distant from the centre, whereas the longitude of aphelion (for 1587) is 148° 48′ 55″. How exactly these elements represent the data, and so may be used to compute exact longitudes as seen from the sun, may be seen from the Table, where the remaining differences between observation and computation are given in the 5th column. "So I state, that the places of opposition are rendered by this computation with the same exactness as Tycho's sextant observations are exact which, through the considerable diameter of Mars and the insufficiently known refraction and parallax, are affected by some uncertainty, surely as much as 2′."

Thus Chapter 18 closes. And then Chapter 19 begins with the words: "Whoever could think it to be possible? This hypothesis so well in accordance with the oppositions, yet is wrong." Ptolemy in bisecting the eccentricity was right. This appears at once when the latitudes at opposition are used, which though not quite are yet sufficiently accurate for the purpose. The ratio of the computed inclination as seen from the sun and the observed value seen from the earth, is the ratio of the distances; making this computation for aphelion (1585) and perihelion (1593) Kepler finds the eccentricity of the sun between 0.080 and 0.0994, at variance with the value found from the oppositions. Computing, on the other hand, the oppositions in the case of a bisection of the total eccentricity, he finds for 1582 a longitude of 107° 4¾′; deviating nearly 8′ from the previous computation and 9′ from observation.

"From this so small deviation of 8′ the cause is apparent why Ptolemy could be content with bisecting the eccentricity. . . Ptolemy did not claim to reach down beyond a limit of accuracy of ⅙° or 10′ . . . It behooves us, to whom has been given by divine benevolence such a very careful observer in Tycho Brahe, in whose observations an error of 8′ of Ptolemy's computation could be disclosed, to recognize this boon of God with thankful mind and use it by exerting ourselves in working out the true form of celestial motions. . . Thus these single 8 minutes indicate to us the road towards renovation of the entire astronomy; they afforded the materials for a large part of this work."

The demonstration is made more rigid by taking other observations at such times as Mars was in aphelion and perihelion, in addition to the oppositions. Here for the first time the trigonometric determination of distance is made use of, which will remain the backbone of his work. In the triangle Sun-Mars-Earth the direction of each side is known: Earth-Sun through observation of the sun (laid down in Tycho tables), Earth-Mars through observation of Mars, and Sun-Mars from his elements. Then from the angles known the ratio of the sides can be computed; taking the distance Earth-Sun from Tycho's tables, which though not entirely good are sufficient for the purpose, he finds the distance Sun-Mars. From these distances again the eccentricity of the sun's position is found to be comprised between 0.08377 and 0.10106, about half the total eccentricity. So he stands before the enigma that computation by the elements, which represent the oppositions perfectly, cannot be true—further on he calls them the vicarious hypothesis — whereas the true values do not represent the oppositions. And he concludes by saying: "Thus what we first had built out of Tycho's observations we had to demolish afterwards because of other observations of the same observer; this by necessity happened to us because on the authority of earlier masters we followed some probable but in reality wrong course. So great pains had to be taken by imitating the earlier masters."

Kepler then gives considerations how a wrong hypothesis can give good results. What really is the matter, we can easily see when using modern analysis that develops the irregularities in longitude in a series of increasing powers of the eccentricity. Since $e = 0.10$ corresponds to 5¾°, $e^2 = 0.01$ to 34′, $e^3 = 0.001$ to 3′.4 of arc, we have only to consider the first and second power; then in well-known notation $v - M = 2e \sin E + \frac{1}{4} e^2 \sin 2E$. In Kepler's hypothesis of a circular orbit with the centre at distances e_1 and e_2 from equant and sun we have $v - M = (e_1 + e_2) \sin E + \frac{1}{2} (e_2^2 - e_1^2) \sin 2E$. Taking $e_1 + e_2 = 2e$ we will have the right result at $E = 90°$ whatever the separate values. To make both expressions identical we have to take $e_2^2 - e_1^2 = \frac{1}{2} e^2$, hence $e_2 = 9e/8$ and $e_1 = 7e/8$, *i.e.*, for $2e = 0.1856$, $e_2 = 0.1044$, $e_1 = 0.0812$, somewhat less different than Kepler's values. For $e_1 = e_2$ the second term disappears, and in the octants, at $E = 45°$, the error is $\frac{1}{4}e^2$, corresponding to 7.5′.

IV

Since uncertainties in the orbit of the earth spoiled the exactness of his computations on Mars, Kepler now turns to a closer examination of the former. It was necessary, besides, because with Ptolemy and Tycho the earth had no equant, whereas as a planet it should not be different from the other ones. He applies the trigonometric method to observations where Mars occupied the same place, the earth different places, in their orbits; thus he found an eccentricity 0.01837 (the five decimals are not an indication of precision but a consequence of Kepler taking always the radius 100,000). Since the fluctuations in angular velocity afforded to Tycho an eccentricity 0.03584, twice as large, it appears that the earth too has an equant. This enables him to construct tables giving the exact distances and longitudes of the sun. But at the same time it leads him to physical and philosophical considerations.

At aphelion, at large distance from the sun, a planet moves more slowly, at perihelion it moves more rapidly, exactly in inverse proportion to the distance. Distance is primary, cannot be dependent on velocity, but velocity must be an effect of distance. The force determining the velocity cannot have its seat in the planet, but proceeds from the central body, the sun. After the lever-principle a planet at greater distance is moved by the solar force with greater difficulty, hence needs more time to describe a certain arc. This physical explanation shows, Kepler says, how he was right to relate all motions to the bodily sun instead of to a void point; and at the same time how Tycho was wrong in having the heavy sun describe an orbit about the earth. The sun is not only the source of light and heat for the entire planetary system but the source of force also. Light and force, both immaterial, expand in space. Since light expands over spherical surfaces, going upward and downward, to all sides, it decreases with the second power of distance. Solar force, on the contrary, expands along circles in the ecliptic, not upward and downward, driving the planets in longitude only through the zodiac, hence decreases with the distance itself. We can understand this solar force, if we assume that the sun rotates about an axis and thus draws along the planets in the same direction, more slowly as they are farther distant. If one asks after the nature of this force he has to give attention to the magnetic force. It is a directing force, as if the magnet consists of threads or fibres; the sun too does not attract the planets—else they would fall into the sun—but directs their course through a sideway force as if it consists of annular magnetic fibres. This is more than an analogy, since Gilbert has found that the earth, which in the same way directs the moon in its orbit, is a magnet.

It would not be right for a modern astronomer to consider these discourses as vain phantasmas. Kepler was not an arid computer, set on simply representing the celestial motions in rows of exact numbers. What mattered to him was physical understanding of things. Not incidentally he put as the full title of his book "Astronomia nova aitiologetos seu physica coelestis" (New astronomy causally explained or celestial physics). His speculations are of the same kind as later on in the 17th century often came up, product of new inquisitive impulses, far superior to the sterile scholasticism of the preceding century that still occupied the academic chairs. On account of his speaking on a force proceeding from the sun he sometimes, wrongly as we see, has been called a precursor of Newton; he was rather a precursor of the natural philosophy of the 17th century. What appears in Descartes' vortex theory as a broad and vague philosophical speculation, in Kepler's ideas has the freshness of direct precise conclusions imposed by the facts of experience.

Whilst the sun produces the general circular motion, it is the planet itself that is responsible for its special irregularity, the alternation of smaller and greater distance. In following up these trends of thought Kepler's ideas often become vague and contradictory; thus he speaks of the spirit or essence of the planet that has to attend to the apparent size of the sun. More important it is that he develops here a new method of computation. The equant as a temporary implement has to be discarded and replaced by the dependence of the velocity on the distance from the real sun. The time spent on a small arc of the orbit is proportional to the distance; so, to have the total time for a longer arc, all the intervening distances have to be summarized. This is a problem of integration—"if we do not take the sum total of all of them, of which the number is infinite, we cannot indicate the time for each of them"—which he first solves by numerical summation. But then he substitutes for the sum total the area between the limiting radii, though this is not exactly the same thing; the area can easily be computed as a circular sector diminished by a triangle. This affords the "physical part" of the inequality; the "optical part" results from that the arc described is viewed from the sun instead of from the centre.

V

Kepler now returns in the 4th Book to Mars, and computes more exactly the distance in aphelion and in perihelion each from five observations, when the earth stood in different positions. He finds them 1.66780 and 1.38500 radii of the earth's orbit, with longitude of aphelion $148° \, 39' \, 46''$. Then the radius of Mars' orbit is 1.52640 and the eccentricity $0.1414 : 1.5264 = 0.09264$, almost exactly half the total eccentricity 0.18564. Computing now for the octant (E = 45°)* to which belongs a "mean anomaly" proportional to the elapsed time, of 41° 14′ 48″, he finds the "adjusted anomaly" (our "true anomaly") to be 49° 0′ 35″. The vicarious hypothesis gives 48° 52′ 34″, so again, as could be expected, we have the old difference of 8′. He now recomputes the triangles formerly used for the distance of the earth, now, conversely, for the distances of Mars; at longitudes 44°, 185°, 158° he finds 1.4775, 1.6310, 1.66255. Computation by means of the circle with the above elements affords 1.48539, 1.63883, 1.66605. Observation thus shows the distances of Mars from the sun to be smaller than what follows from the circular orbit. And the differences cannot be ascribed to chance errors. "I speak to you, expert astronomers who know that evasive sophistries, so common in other sciences, in astronomy are available to none." Sideways the planet took its course within the circle described through aphelion and perihelion. "The matter is obviously this: the planetary orbit is no circle; to both sides it goes inward and then outward till in the perigee the circle is reached again. Such a figure is called an oval."

And now at once he sees the way out of the contradictions through the following explanation. An eccentric circle can be considered as a centric circle combined with an epicycle described backward in the same time, whereby the epicycle's radius keeps the same direction in space. Now the circle produced by the sun's force is described more rapidly where the distance is small; the epicycle being the action of the planet itself, keeps a constant velocity. So from perihelion onward the angles described by the epicycle's radius are smaller than those described by the radius from the sun; the result is, as shown by Figure 1, that the planet gets inside the eccentric circle. Kepler prefers another form of construction for this idea; when the planet from perihelion A (in Figure 2) through the stronger solar force has been propelled in direction as far as the radius SP_1 it has changed by itself its distance not more than corresponds to the smaller way from A to B_1 (B_1 determined by arc AB_1 and area CAB_1 = area SAP_1 proportional to the time); making $SP_1 = SB_1$ we see that its place P_1 is situated within the circle. At an anomaly E of 90°, where the distance SP_2 is equal to SB_2, the figure can show that the point P_2 is situated an amount of e^2 (0.00858 for $e = 0.09264$) inside the eccentric circle.

The computation, however, now becomes extremely troublesome; the resulting oval is somewhat egg-shaped, and to compute the areas we have first to know the unknown places. In the case of an ellipse the areas could be computed easily because elliptic sectors are proportional to circular sectors. Since the deviations of the oval from an ellipse are

*Kepler, following the old use, with Ptolemy and Copernicus counts all his anomalies from the aphelion. We have taken them, after modern usage, from perihelion; hence the values given here for the first octant 45° are the supplements of Kepler's values for 135°.

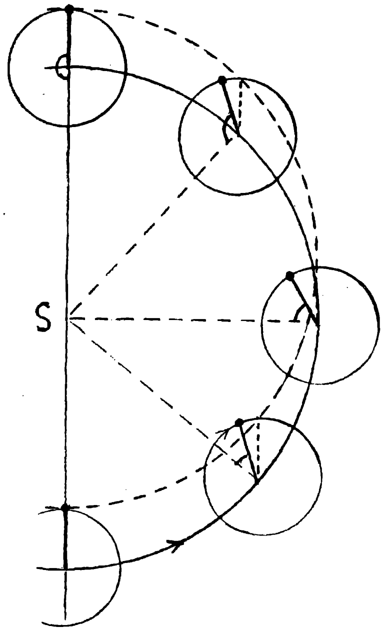

FIGURE 1

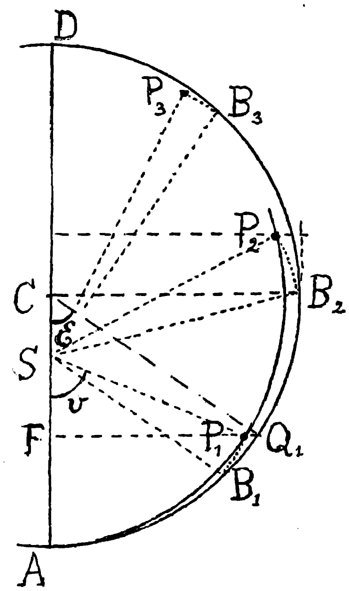

FIGURE 2

not large, Kepler decides to neglect them and make his computations as if it were an ellipse. His computation, if we express it in modern notation, comes down to the following formulas

$$\frac{\text{time AP}}{\text{per. rev.}} = \frac{\text{M (Mean an.)}}{2\pi} = \frac{\text{area ACB} = \text{ASP}}{\text{area ellipse}} = \frac{\text{area ASQ}}{\text{area circle}} = \frac{\tfrac{1}{2}\,(\text{E} - e\sin\text{E})}{\pi},$$

hence $\text{M} = \text{E} - e\sin\text{E}$. Then the angle $\text{ASP} = v$ is found by the geometrical relation

$$\text{tang. } v = \text{FP/SF} = (1 - 0.00858)\ \sin\text{E}/(\cos\text{E} - e).$$

For $\text{E} = 90°$ the computation leaves only insignificant differences. For $\text{E} = 45°$ he finds $\text{M} = 41°\,14'\,48''$ and $v = 48°\,45'\,55''$, whereas the vicarious hypothesis gave $48°\,52'\,34''$, nearly $7'$ larger. Thus again it is wrong in the octants, as much as before, but now in the opposite direction. Carefully he scrutinizes whether the cause of the difference could be detected in his approximate methods and suppositions; but nothing is found. And now he tells the reader that indeed he has got on a wrong track. "As soon as Brahe's most precise observations had taught me that the orbit was no exact circle but to both sides curved more inwardly I at once believed I knew the natural cause of this deviation. . . Because for eagerness I was blind. . . I stuck to the first cogitation offering itself that looked so wonderfully probable on account of the uniformly described epicycle. So I came into a new labyrinth from which we will have to find a way out in this and the next chapters."

Indeed his bent for understandable physical causes had played a trick on him and cost him a couple of years of needless labor. And in his war-metaphoric language he tells how he thought to have triumphed and locked the foe in the fetters of eccentric equations and the prison of numerical tables, but now saw him breaking loose from his prison and resuming his liberty. So a new campaign is necessary.

VI

First he resumes and improves his former computations: longitude of aphelion 149° 0′ 40″ (for 1600), distance in aphelion 1.66465, in perihelion 1.38234, radius 1.52350, eccentricity 0.09265. Comparing now the distances derived from observations far from the apsides with the values computed in the oval orbit, the latter appear to be too small, nearly 0.00660. The oval is situated too much inside, nearly the same amount as the circle was too much outside relative to the places derived trigonometrically. The real orbit must be situated nearly in the midst between the circle and the oval, what corresponds to the computation of the octants where circle and oval also deviate an equal amount to different sides. The maximal breadth of the deficient sickle is not 0.00858 but the half of it, 0.00429. "When I pondered ever again over the fact that my triumph over Mars had been futile, my eye fell upon the secant of 5° 18′, which is the maximal optical inequality. When I perceived it to be 1.00429 it was as if I awoke from sleep and saw a new light."

The breadth of the sickle cut off at both sides from the circle is not e^2 but $\frac{1}{2}e^2$. In an ellipse with eccentricity e half the short axis is $\sqrt{1-e^2}=1-\frac{1}{2}e^2$. The sun thus occupies the focus of the ellipse; Kepler does not state it expressly but it is self evident now. Again he tries to find physical explanations: that planetary magnetism makes the attraction larger when one of the poles is inclined toward the sun, and smaller for the other pole; but this does not fit in with the case of the earth. Or a planetary spirit is assumed that feels out the distance of the sun or the described road; but here the arbitrariness of behavior is too great. Kepler as a phantastic thinker always tests his phantasies critically by means of empirical data.

Now the mathematics of the ellipse are worked out. In consecutive geometrical lemmas he demonstrates that in this elliptic orbit the distances from the sun vary just as if the planet with regularly increasing E moved regularly up and down on the diameter of an epicycle—our modern equation $r = a\,(1 - e\cos E)$. Then, that the summation of all the distances for regularly increasing E results in precisely the area of the sector swept over by the radius—we say that integration of $r = a\ (1 - e \cos E)$ produces $E - e \sin E$. Extensively he shows that all the neglected small differences in former approximations cancel out exactly in the elliptic motion. He excuses himself for the difficulty of these demonstrations. "I would not leave these things untouched though they are very difficult and not very necessary for the practice of astrology, which many consider the only goal of celestial philosophy." And he advises to read Apollonius. Then he expounds the formulas for computation, the same as given already above (with the change only of the numerator factor $1 - 0.00858$ into $\sqrt{1 - e^2} = (1 - 0.00429)$; but what was a make-shift there now is the sure basis of reality.

For given eccentric anomaly E it all goes smoothly and easily; but not if we have to make the computation for a given time. The finding of E from $E - e \sin E = M$, afterwards called Kepler's equation, is recognized and put forward by himself as a difficult problem not solvable by the ordinary methods. "If the mean anomaly is given no geometrical method exists to find the adjusted and the eccentric anomaly. For the mean anomaly is composed of two areas, a sector and a triangle. . . If their sum total is given, we cannot tell what is the value of the arc and what of the sine belonging to this sum, if we have not ascertained beforehand what area belongs to a given arc, *i.e.,* if we have not constructed tables to work with them afterwards. Such is my opinion. The less handsome this method may look geometrically, the more I exhort the mathematicians to solve for me the following problem: . . . to divide in a given proportion the area of a semicircle by a line from a given point of the diameter. I am content with my conviction that a solution *a priori* is not possible on account of the dissimilarity of arc and sine. Whoever will show me my error and a way out will be to me as great as Apollonius."

In the last chapters there remains to derive the situation of the plane of the orbit. He finds for the longitude of the ascending node $46° 46\frac{1}{3}'$ and for the inclination $1° 50' 25''$. The latitudes observed in the different oppositions are, as is shown in the table, well represented by these elements. The remaining deviations are larger than for the longitudes, due mostly to uncertainties in parallax and refraction. Kepler points out that a parallax of Mars larger than some few minutes would spoil this concordance. Finally the observations of Ptolemy are used to determine the secular variation in the line of apsides and the line of nodes.

VII

Something must be said, besides, on the introduction printed in the front but undoubtedly written after the work was finished. Here Kep-