The Planetary Theory of Laplace

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I

The main problem occupying astronomers and theorists in the 18th century was this, whether Newton's law of gravitation would be able to explain the motions of the heavenly bodies. Newton had derived the law, so extremely simple in its mathematical form, from Kepler's laws of the planetary orbits; and he had confirmed its validity by explaining in this way the precession of the equinoxes, the tides, and some of the most notable irregularities of the moon. But it implied that Kepler's laws could not be strictly true; the mutual attraction of the planets produces perturbations of their regular courses, some of which had already become evident through the increasing accuracy of the observations. So the problem was to give an explanation of these irregularities by means of theory. Or, in a more general way, theoretically to derive with the utmost precision the motions of the planets and the moon, and to compare them with the observational results. A group of the most brilliant theorists came forward, mostly in France, such as Clairaut, Euler, d'Alembert, and later Lagrange and Laplace, who set themselves this task. While gradually the methods of computation developed under their hands, the explanation of such riddles as the secular acceleration of the moon and the gradual opposite change of the periods of revolution of Jupiter and Saturn, for a long time baffled the most astute mathematicians; and only in the later part of the century Laplace succeeded in giving a solution.

All this progress was made possible only owing to the introduction of a new form of mathematical treatment. Algebra, calculus, had to replace geometry. It was well known how geometrical problems which by their complexity demanded the keenest insight and perspicacity, could be solved in a simple way, by downright computation, when transformed into algebra by the methods of analytical geometry. The still more difficult problems of mechanics could be solved by calculus only. Celestial mechanics, the science of the motions of the celestial bodies on the basis of Newton's law of gravitation, could develop in the 18th century only by abandoning Newton's geometrical treatment and converting it into an algebraic computation. This was the work of the 18th century scientists already named. The final result of their work at the end of the century was laid down in Laplace's *Traité de Mécanique Céleste* (Treatise of Celestial Mechanics).

In his introductory "Plan of the Work" he writes: "Newton published, towards the end of the past century, the discovery of the universal gravitation. Since that time the geometers have succeeded in reducing to this great law of nature all known phenomena of the world system, thus giving an unexpected precision to the astronomical theories and tables. . . Astronomy, considered in its most general way, is a large problem of mechanics, in which the elements of the motions appear as the arbitrary constants; its solution at the same time depends on the exactness of the observations and on the perfection of the analysis, and it is of the highest importance to remove all empiricism and to take from observation only the indispensable data. . ." As a curious remembrance of the time of the publication (1799) the last sentence of the preface may be quoted: "I will adopt here the decimal division of the right angle and of the day, and all linear measures will be expressed in the *mètre,* determined from the arc of the terrestrial meridian between Dunkirk and Barcelona." Whereas the length unit with its decimal system has maintained itself in science, the decimal division of angles and times has not succeeded in coming into use; thus expressed in his seconds of time of 0.864 of our seconds, and his seconds of arc of 0″.324 we meet here with the uncommon sight of a gravity acceleration of 3.65548 *mètres* and of a lunar parallax of 10536″.

II

From the general analytical treatment of motion and equilibrium, and of systems of bodies attracting one another, which Laplace gives in the first part of his work, we take here the simple formulas needed for the special cases of planetary motions. Analytical treatment of phenomena in space means use of coordinates, rectangular xyz, or other ones. Newton's law then says that the acceleration, the second differential quotient of a coordinate is given by the force in that direction. Laplace introduces here the space function Q—afterwards called potential—the gradient of which in any direction is the force in that direction. Hence

$$d^2x/dt^2 = P_x = \partial Q/\partial x, \text{ similarly for } y \text{ and } z.$$

With the sun (mass M) as the only attracting body we have for the planet at distance r

$$P = -M/r^2;\ Q = M/r;\ d^2x/dt^2 = -M/r^2 \cdot x/r, \text{ similarly for } y \text{ and } z.$$

First the problem of two bodies, the simple elliptic motion of a planet, is discussed. Whereas Newton solved in a rigid and elegant way the problem, how to derive the law of attraction from the given elliptic motion, he did not demonstrate explicitly the more difficult problem that in the case of the inverse square law the orbit needs must be a conic section. Laplace, on the contrary, through the analytical method easily solves the latter problem by a double integration; this demonstration has since found its way into every textbook.

In the case of more attracting bodies their forces have to be added together. Here the usefulness of the potential function Q presents itself; forces must be combined geometrically because (being vectors) they have different directions, but the space functions of which they are the gradients are numerical values (scalars) and can be added algebraically. Thus for each disturbing planet a term is added. Or, more precisely, two terms. Because the solar mass surpasses all the planetary masses by factors of more than 1000, the effect of the planetary attractions appears as small secondary corrections to the primary regular orbit. It is these corrections, the perturbations, for which equations are formed, and the additional disturbing forces that produce them are the gradients of an additional potential function, called the "disturbing function" R. Since the resulting motion is referred to the sun as the centre of coordinates, the effect of the disturbing body, the change in place relative to the sun, is determined by what its attraction upon the planet is different from its attraction upon the sun; hence the "direct" term expressing the attraction of the planet, has to be diminished by an "indirect" term expressing the attraction of the sun by the body. Thus for one disturbing body, with mass m', distance r' from the sun and ρ from the disturbed planet, and with θ the angle between the radii r and r',

$$d^2x/dt^2 = -Mx/r^3 + \partial R/\partial x, \text{ etc. } R = m'/\rho - m'rr' \cos\theta/r'^3.$$

In the case of more disturbing planets for each of them two analogous terms appear in R.

Laplace transforms these equations for the rectangular coordinates into equations for the polar cordinates: the distance, the longitude, and the deviation perpendicular to the plane of the orbit. These equations have to be integrated.

III

The differential equations, having the required unknown coordinates themselves contained in the expressions of the forces and of the disturbing function, are far too complicated to be solvable in a direct way. They must be solved by successive approximations; first the undisturbed values of the coordinates are substituted in the expressions of the forces, and by thus integrating the equations the resulting changes of the coordinates are found. They are of the order of the disturbing masses (the largest of which is 1/1047) and are called perturbations of the first order. Then these changes should be introduced in the forces, giving small deviations in the equations and producing deviations in the first-found results; these are the perturbations of the second order with respect to the masses. Because they are, at most, a thousand times smaller than the first-order results, they usually can be neglected; only in exceptional cases it is necessary to give attention to second-order terms.

But the equations for the first-order perturbations also are too complicated to be solved in a direct way. The method developed by 18th century analysis consists in decomposing each function occurring in the expressions into an infinite series of goniometric functions, which each of them is easily tractable. In the undisturbed elliptic motion the radius as well as the anomaly are periodic functions which can be expressed in a series of terms containing sines and cosines of multiples of angles increasing uniformly with the time, at a rate given by the mean angular motion n of the planet. With ϵ the longitude at $t = 0$ and π the longitude of perihelion, it is multiples of the mean anomaly, the angle $nt + \epsilon - \pi$, which appear in all the expressions for radius and longitude. Thus

$$r/a = (1 + \tfrac{1}{2} e^2) - e\,(1 - \ldots) \cos\,(nt + \epsilon - \pi) - \tfrac{1}{2} e^2\,(1 - \ldots) \cos 2\,(nt + \epsilon - \pi) - \text{etc.}$$

$$l = nt + \epsilon + (2\,e + \ldots)\,\sin\,(nt + \epsilon - \pi) + (5/4\,e^2 - \ldots) \sin 2\,(nt + \epsilon - \pi) + \text{etc.}$$

Each coefficient contains a series of terms of powers of the eccentricity increasing by 2, and the lowest power is equal to the coefficient before the angle. Moreover the angular distance of two planets θ, that appears as the chief variable determining the variations in mutual distance and mutual disturbing force, is given, in the case of coinciding planes of the orbits by $(n' - n)\,t + \epsilon' - \epsilon$.

In what way the direct term of the disturbing function depends on this angle is indicated (for the case of Jupiter and Saturn, where the ratio of the mean distances to the sun is 0.5453) by the heavy line in Figure 1. This function has to be resolved into a series

$$1/\rho = \tfrac{1}{2} A_0 + A_1 \cos\,\theta + A_2 \cos\,2\theta + A_3 \cos\,3\theta + \text{etc.}$$

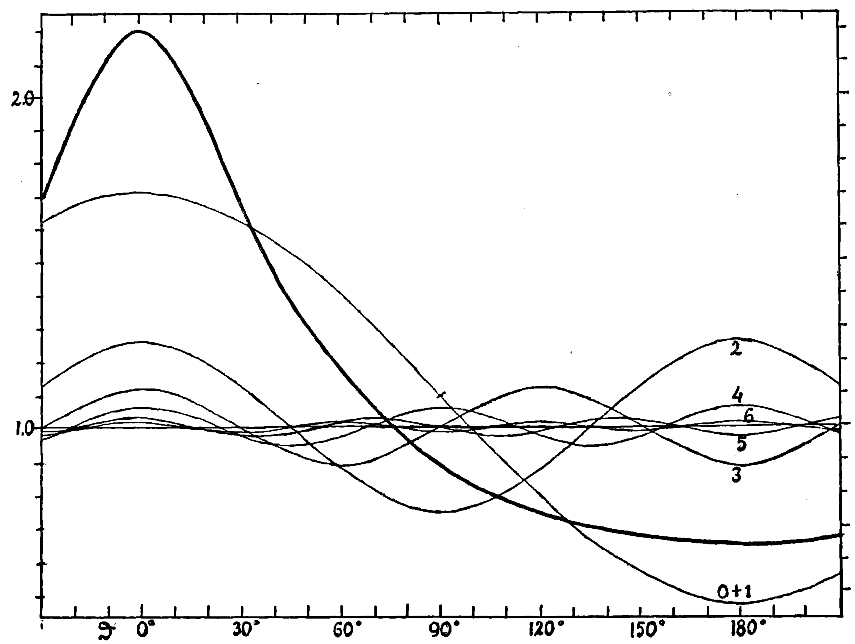

FIGURE 1
DECOMPOSITION OF THE DISTURBING FUNCTION

Introducing $a/a' = \alpha$ (always the larger radius in the denominator):

$$1/\rho = 1/\sqrt{(a^2 - 2aa' \cos\theta + a'^2)} = 1/a' \ (1 - 2\alpha \cos\theta + \alpha^2)^{-\frac{1}{2}},$$

it comes down to a development of the latter function. Laplace devised a most elegant development of the more general case where the exponent is taken $-s$ and s can have the values 1/2, 3/2, 5/2, etc. It is given in the form

$$(1 - 2\alpha \cos\theta + \alpha^2)^{-s} = \tfrac{1}{2} b_s^{(0)} + b_s^{(1)} \cos\theta + b_s^{(2)} \cos 2\theta + \text{etc.}$$

where the coefficients b are since known as "Laplace's coefficients." They are functions of the ratio α; Laplace developed them into power series, studied their properties, and indicated how only the first members have to be computed directly, whereas all the others then can be derived easily by means of their mutual relations. By means of the numerical values of all these coefficients which he gives in his 6th Book —in our case $b^{(0)} = 2.1802$ and the other b's are 0.6206, 0.2576, 0.1180, 0.0566, 0.0278, 0.0139, 0.0070, etc.—we have inserted the separate component curves into our Figure, the 2nd down to the 6th as oscillations about the line 1.0. The coefficients b for $s = 3/2$ and the derivatives with respect to α, which are needed for the case of eccentric or mutually inclined orbits, are likewise computed.

In the case of circular non-inclined orbits the disturbing function is restricted to the series in the elongation θ, where the indirect term is included in the second term with coefficient A_1. In the case of elliptical orbits the factor r/a and the difference $l - (nt + \epsilon)$ give rise to products of goniometric functions. These products are easily reduced to sums of such functions, whereby sums and differences of the arcs appear as arguments. So for instance

$$A_3 \cos 3\ (n't - nt + \epsilon' - \epsilon) \times e \cos\ (nt + \epsilon - \pi) =$$
$$\tfrac{1}{2} A_3\, e\ \{\cos\ (3n't - 2nt + 3\epsilon' - 2\epsilon - \pi) + \cos\ (3n't - 4nt + 3\epsilon' - 4\epsilon + \pi)\}$$

In the same way r' of the other planet gives rise to terms with the coefficient e'. When the multiplicity of nt and $n't$ (as well as of ϵ and ϵ') are different by 1 the coefficient contains e or e' in the first power; when by 2 or 3 it is of degree 2 or 3 in the eccentricities. In the same way the mutual inclination of orbits produces a number of terms having ascending powers of the inclination (more exactly: of tan $\tfrac{1}{2}\phi$) in the coefficients and the longitude of the node ☊ in the arguments. So the disturbing function consists of a large, or rather infinite number of terms which, however, since the Laplace coefficients b rapidly decrease and the eccentricities are of the order 0.1 or smaller (Jupiter 0.048, Saturn 0.056, Mars 0.093, only Mercury 0.206), have only to be used in a limited number.

All these terms appear in the disturbing force; its irregular and complicated variation has been dissolved now into a large number of entirely regular periodic forms, sines and cosines of arcs containing the time multiplied by a certain multiple of n' and a certain multiple of n, say $i_1n' - i_2n$. Then the changes in the coordinates are found by double integration with respect to the time. This integration, for a goniometric function, simply comes down to division of the amplitude by the square of the coefficient of t, hence by $(i_1n' - i_2n)^2$. So the perturbations in the coordinates also are found as a series of terms of the same kind.

Laplace instead of in rectangular coordinates derives the more natural perturbations in radius and longitude δr and δv, for which the disturbing force is somewhat more complicated, and the equations take a different form. He introduces the simple periodic function $u = e \cos (nt + \epsilon - \pi)$ on which in the undisturbed elliptic motion the variations in radius and in longitude or anomaly depend by means of simple relations expressible in rapidly converging series. The same relations between the disturbed u and the disturbed radius and longitude hold in the real motion; hence the differential equation for u with all the perturbation terms has to be derived and solved. This equation has the form

$$d^2u/dt^2 + n^2u = \Sigma\, p \cos (i_1n't - i_2nt + \ldots)$$

where the second member consists of all the perturbation terms as derived and mentioned above. Without these terms the solution would be a simple "free" oscillation with frequency n and arbitrary amplitude and phase [such as $e \cos (nt + \epsilon - \pi)$]; now "enforced oscillations" are added, having the periods of the additional forces, one for each term

$$u = C_1 \cos nt + C_2 \sin nt + \Sigma\, [p/\{(i_1n' - i_2n)^2 - n^2\}] \cos (i_1n't - i_2nt + \ldots)$$

From this δu consisting of the sum total Σ, the perturbations in radius and longitude δr and δv are derived and consist of analogous terms.

In this way Laplace has solved the problem, first theoretically in his 2nd Book, and then, in his 6th Book, in numerical values. In the case of Jupiter as disturbed by Saturn, *e.g.*, we find in the perturbations in longitude first a series of terms without eccentricity:

$$+81''\!.2 \sin \theta - 204''\!.4 \sin 2\theta - 16''\!.9 \sin 3\theta - 3''\!.9 \sin 4\theta \ldots - 0''\!.041 \sin 9\theta$$

where θ stands for $n't - nt + \epsilon' - \epsilon$ (Laplace gives 6 decimals of his smaller seconds); then 22 terms follow with the first powers of the eccentricities e or e', among which the largest are

$$-138''\!.4 \sin (2, 1, \pi) + 56''\!.6 \sin (2, 1, \pi') - 44''\!.4 \sin (3, 2, \pi) + 84''\!.9 \sin (3, 2, \pi')$$

where $(3, 2, \pi)$ means $3n't - 2nt + 3\epsilon' - 2\epsilon - \pi$; Laplace gives them down to $(7, 6, \pi)$. Then follows a series of terms of the second degree in e and e', of course not complete. "The large number of inequalities depending on the squares of the eccentricities and the inclinations forbids to compute all of them; in our choice we are guided by the following considerations." These considerations relate to the occurrence of small coefficients of t or coefficients nearly equal to n, which in integrating produce large amplitudes. In the same way the perturbations of Jupiter by the other planets are given, in a smaller number of terms, because they are smaller; the largest term produced by the earth is 0″.12, by Uranus 0″.05. The perturbations of Saturn caused by Jupiter are of course larger. Among Laplace's numerical results we find as the most important terms

$$+3''\!.2 \sin\theta - 31''\!.5 \sin 2\theta - 6''\!.6 \sin 3\theta - 2''\!.0 \sin 4\theta \ldots - 182''\!.1 \sin(1, 2, \pi') + 417''\!.1 \sin(1, 2, \pi) + 34''\!.3 \sin(2, 3, \pi') - 17''\!.7 \sin(2, 3, \pi),$$

and among the terms depending on the squares of the eccentricities there is one amounting to —669″.7 with $(2n' - 4n)\,t$ in the argument. Uranus produces here terms

$$+9''\!.2 \sin\theta - 14''\!.5 \sin 2\theta + 25''\!.2 \sin(3, 2, \pi').$$

The motion of the earth shows inequalities 5″.3 sin θ +6″.0 sin 2θ and 3″.7 sin $(2, 1, \pi)$ due to Venus, 3″.5 sin 2θ due to Mars, 7″.1 sin θ due to Jupiter, in total 47 terms. So at the close of these computations Laplace says: "It suffices to remark here that before the discovery of these inequalities the errors of the best tables reached 35 or 40 minutes [we remember that his minutes were only 32″.4] and that they now do not surpass one minute. . . I have reason to believe that the preceding formulas computed with special care, will add a new precision to the tables of the motions in the planetary system."

Laplace's work was not only a comprehensive summary of what the 18th century had elaborated in exact computation of the planetary motions; it was at the same time the starting point for new progress in the next century. The exactness reached by his formulas, diminishing the errors to some few tens of seconds, was not adequate to the increasing exactness of the observations. The theorists of the 19th century built further upon the foundations laid by Laplace; more refined methods of analysis were devised, terms of higher order were considered and their number was increased. Bessel's functions, Cauchy's numbers, Hansen's product-series made easier the handling of elliptic developments. Leverrier who devoted his life to the exact computation of the planetary perturbations, included terms of the 7th order in eccentricity and inclination; thus the uncertainty of the tables and their deviation from observation went down to a few seconds of arc. Then a more general treatment consisting in a surveyable organization of manipulations up to the highest orders by means of his differential operators enabled Newcomb to establish in full completeness the theoretical system of planetary perturbations.

IV

One of the main puzzles of 18th century theory was the great inequality of Jupiter and Saturn. Halley had stated, and introduced (1695) in his tables, that Jupiter since many centuries was accelerating and Saturn was retarding. Should this go on in the same way, Jupiter approaching to the sun ever more, the stability of the solar system was endangered. The Paris Academy in 1748 and again in 1752 offered a prize as reward for a good solution of the problem, but no satisfactory answer was received. Lambert perceived in 1773 that in the later years Jupiter was retarding, Saturn accelerating, so that it must be a periodic phenomenon. At last Laplace succeeded in finding (1784) the true explanation; and the treatment of this large mutual perturbation of the two planets occupies an important place in his Treatise.

It was the discovery of the importance of long-period inequalities in the planetary motions and of their origin from the occurrence of small divisors in the integrations. "It is especially in the motion of Jupiter and Saturn, the two largest bodies among the planets, that the mutual attraction of the planets is sensible. Their mean motions are nearly commensurable, since 5 times the motion of Saturn is nearly equal to 2 times that of Jupiter; the considerable inequalities arising out of this relation, the laws and the cause of which were unknown, for a long time seemed to make an exception to the law of universal attraction, and now are one of its most striking demonstrations. It is extremely interesting to see with what precision the two chief inequalities of these planets, the period of which comprises nine hundred years, satisfy the ancient and modern observations; the coming centuries will show this concordance ever more in their further development."

Thus, among the innumerable terms of different orders our attention is directed to those containing the sine or cosine of $5n't - 2nt + \ldots$ What is their period? The mean daily motion of Saturn is 120″.455, of Jupiter 299″.128, hence $5n' - 2n = 4''.02 = n/74$ (Jup.); the entire circle is performed in 323,000 days = 74 Jupiter periods = 887 years. If, then, in the disturbing function a term occurs of this form:

$$d^2\,(\delta v)/dt^2 = Cn^2 \sin\{(5n' - 2n)\,t + \ldots\}$$

double integration affords a term in the perturbation in longitude

$$\delta v = -C\,[n^2/(5n' - 2n)^2]\,\sin\{(5n' - 2n)\,t + \ldots\}$$

Such a term in the disturbing force, through the small divisor, is increased more than 5000 times in the resulting longitude of the planet. A small acceleration working ever again in the same way for a long time produces a notable velocity, and this velocity during the same long time results in a large displacement. Thus terms in the disturbing function which are extremely small because they contain high powers of the eccentricities, can yet give rise to perceptible terms in the longitude. But here they are not extremely small. It is the 3rd power already of the eccentricities that appears in the coefficients of the terms in question; so the result is such a large perturbation as to alarm the astronomers during an entire century. Indeed we have such products as $e^3 \cos 3\,(nt + \epsilon - \pi) \times \sin 5\,(n't - nt + \epsilon' - \epsilon)$ producing $e^3 \sin (5n't - 2nt + 5\epsilon' - 2\epsilon - 3\pi)$, $e^2e' \cos 2\,(nt + \epsilon - \pi) \cos (n't + \epsilon' - \pi') \times \sin 4\,(n't - nt + \epsilon' - \epsilon)$ producing $e^2e' \sin (5n't - 2nt + 5\epsilon'$

$-2\epsilon - 2\pi - \pi'$), $e'\gamma^2 \cos\ (n't + \epsilon' - \pi') \cos 2\ (nt + \epsilon - \Omega \times \sin 4\ (n't - nt + \epsilon' - \epsilon)$ producing $e'\gamma^2 \sin\ (5n't - 2nt + 5\epsilon' - 2\epsilon - \pi' - 2\Omega)$, where γ stands for tan $\frac{1}{2}\phi$. Thus Laplace in his development of the disturbing function finds six terms having the third order quantities

$$e^3,\ e^2e',\ ee'^2,\ e'^3,\ e\gamma^2 \text{ and } e'\gamma^2$$

in their coefficients, which have $(5n' - 2n)\ t$ in their argument, and thus after integration have $(5n' - 2n)^2$ as divisors. Taking them all together the result is

$$\text{for Jupiter } \delta v = 1263''.8 \sin\ (5n't - 2nt + 5\epsilon' - 2\epsilon) + 119''.5 \cos\ (5n't - 2nt + 5\epsilon' - 2\epsilon),$$

$$\text{for Saturn } \delta v = -2931.1 \sin\ (5n't - 2nt + 5\epsilon' - 2\epsilon) - 223''.2 \cos\ (5n't - 2nt + 5\epsilon' - 2\epsilon).$$

Thus Jupiter deviates up to 21′, Saturn up to 49′ from their regular places.

Laplace does not content himself with having explained this conspicuous irregularity. Terms of the 5th order in the eccentricity, which may be expected to be nearly 400 times smaller, will be very perceptible and have to be included into an exact derivation of the amplitudes. Terms of the second order with respect to the masses, which will be nearly one thousand times smaller also may be perceptible and have to be computed. Thus he says, in continuance of the words quoted above: "In order to make the comparison more easy for the astronomers I have extended the approximation up to terms dependent on the square of the disturbing force; so I may hope that the values I assigned to them will be only slightly different from what will be found from a long series of observations continued over an entire period." The resulting total values of the four above coefficients are 1237″.41, 112″.71, 2872″.84, and 261″.01.

The discovery of the origin of this large perturbation has directed the attention of Laplace and later theorists to look at inequalities of long period in the case of other planets too. Thus he found for Venus one of 1″.50 with argument $(5n' - 3n)\ t$ due to the earth, and 2″.01 with argument $(3n' - n)\ t$ due to Mars, and for the earth one of 1″.13 due to Venus. Uranus has an inequality of 132″.51 due to its mean motion being nearly 1/3 of Saturn's, with a period of 7 of its revolutions.

V

In Laplace's treatment of the perturbations in radius and longitude as enforced oscillations in a system oscillating with its proper frequency n, special terms arise in the case that the acting force has the same frequency. In that case (resonance) the force acts in the same way in every oscillation and tends to increase indefinitely the amplitude of the oscillation which, in the absence of resistances, must grow to infinity.

Mathematically this appears in such a way that by integrating

$$d^2y/dt^2 + n^2y = 2\ n\ p\ \cos\ (nt + \alpha)$$

we get

$$y = C_1 \cos nt + C_2 \sin nt + pt \sin\ (nt + \alpha),$$

so that the amplitude contains the factor t. Expressed in another way: the development of the disturbing function gives rise to terms that do not contain the sines or cosines, or have lost the time in their arguments; then by integration t appears as a factor in non-periodic terms. Such is the case with the term e^2 in the development of r/a. Such is the case also in the multiplication

$$A \cos\ (n't - nt + \epsilon - \epsilon') \times e \cos\ (nt + \epsilon - \pi) \times e' \cos\ (n't + \epsilon' - \pi')$$

which produces a term $1/4\ Aee' \cos\ (\pi' - \pi)$ where, if we restrict ourselves to the first power of the masses, $\pi' - \pi$ is a constant.

"The disturbing forces introduce . . . the time t outside the sine and cosine, or in the form of circular arcs which by their indefinite increase must make in the long run these expressions erroneous; hence it is essential to make these arcs disappear. . . As these variations take place with great slowness, they are denoted by the name 'secular inequalities.' Their theory is one of the most interesting parts of the world system; so we will expound it with all the ampleness that its importance demands." Indeed the exhaustive treatment of these secular perturbations by Laplace is a masterpiece of celestial mechanics and deserves the ample place devoted to it in his great work.

The equations determining the variations of the elliptical elements are derived by Laplace in a somewhat intricate way, making use of general considerations on differential equations. This part of the theory has been made far more direct and elegant in the next century by the introduction of canonical equations and canonical elements by Jacobi. In the practical elaboration of the equations, however, and their solution Laplace's treatment has remained classical up to modern times. Instead of the elements e and π he introduces the rectangular components, of e, *viz.*, $e \sin \pi = h$ and $e \cos \pi = l$ — representing rectangular coordinates of the focus, the sun, relative to the centre of the ellipse— and likewise the inclination is decomposed into $\tan \phi \sin \Omega = p$, $\tan \phi \cos \Omega = q$; they are called afterwards the eccentric and the oblique variables. Then the equations determining the secular perturbations of the elements are found to be

$$dn/dt = 0;\ d\epsilon/dt = 0\ \text{(by an appropriate definition of } \epsilon)$$
$$dh/dt = (0, 1)\ l - [0, 1]\ l';\ dl/dt = -(0, 1)\ h + [0, 1]\ h';$$
$$dp/dt = (0, 1)\ (q' - q);\ dq/dt = (0, 1)\ (p - p'),$$

where (0, 1) and [0, 1] represent expressions in the constants A and b and the masses of the two planets denoted by the index 0 and 1.

"The equation $dn/dt = 0$ just found is of great importance in the theory of the world system, since it shows that the mean motions of the celestial bodies and the major axes of their orbits are unalterable; but the equation is approximate only. . . It is highly important to know whether higher powers of the eccentricity should disprove this result and produce terms with the time as factor. We will demonstrate that, with respect only to the first power of the disturbing masses, the expression for v, however far the approximations include higher powers of the eccentricities and inclinations, will not contain such terms." Then Laplace gives a demonstration along general lines that no terms with the time as a factor will occur, provided that the mean motions or the periods of revolution of the planets are not commensurable.

Turning now to the eccentricities we see that in the equations for each planet the time-differential of each component is a linear function of the other component, of itself and of the disturbing planet. Hence, since there were 7 planets at the time, it is a linear function of the other component of itself and of all the other six planets. This holds for every planet, so that we have a system of equations such as

$$dh_1/dt = a_{11}l_1 + a_{12}l_2 + a_{13}l_3 + \ldots + a_{17}l_7$$
$$-dl_1/dt = a_{11}h_1 + a_{12}h_2 + a_{13}h_3 + \ldots + a_{17}h_7$$
$$dh_2/dt = a_{21}l_1 + a_{22}l_2 + a_{23}l_3 + \ldots + a_{27}l_7$$

et cetera, where the a_{ij} are all known constant coefficients. Substituting in the second members the present values of h and l of the planets, considered as constants, we find the yearly variation of these elements, hence of e and π. But the equations can also be integrated analytically so as to give all these elements directly as functions of the time. Laplace shows that there are solutions of the form $h = N \sin(gt + \beta)$; $l = N \cos(gt + \beta)$, the argument $gt + \beta$ being identical for all the planets. They represent slow circular motions of the centre of the planet's ellipse about the sun in a period $2\pi/g$. By eliminating all the amplitudes N it appears that g is determined by an equation of the 7th degree, which has 7 different roots, all real and positive. Hence for any planet there are 7 circular motions of the centre of the orbit about the sun, combining each with its own period. The periods and phases are the same for all the seven planets, but the radii or amplitudes are different. Laplace computed and gives all the coefficients a_{ij} of the equations; but he did not compute, at least does not give, the periods themselves and the radii of the circular motions, probably because, amounting to ten thousands of years or more, they lie far outside any practical application and any test. In their stead he derives and gives the yearly variations of the elements for each of the planets for the year 1750.

Analogous results are found for the secular changes in the components of the inclination p and q. Their representation by a circular function means that the pole of the orbit describes at the celestial sphere a small circle about the pole of the plane of reference. Here also we have 7 circular motions with very long but different periods, with identical periods but different radii for the different planets. Here also Laplace does not give the periods but only the yearly variations of the elements, inclination and node, for 1750.

A special discussion is devoted to the stability of the planetary system. It depends on the character of the roots of the 7th degree equation for g. Should some of them be equal or complex, then the sines and cosines would turn into arcs or into exponentials, so that the eccentricities could increase indefinitely and the stability of the system would be destroyed. Laplace now shows that the sum total of the expressions $e^2m\sqrt{a}$ for all the planets must always remain equal to the same constant value, which for our planetary system is a small quantity. Then each of the terms must remain less than this constant, provided that they all are positive, *i.e.*, that all planets revolve about the sun in the same direction. And he shows that in this case all the roots g are real and different. "Hence the system of orbits is perfectly stable in regard to their eccentricities; the orbits only oscillate about an average state of ellipticity, from which they deviate little, keeping their major axes constant."

The same holds for the inclinations. From the general dynamical principles he had derived already that there exists an "invariable plane," the plane of maximum moment of momentum for the entire system, the natural plane of reference for the separate orbits. Moreover he finds that the sum total of $m\sqrt{a}\ \tan^2\phi$ for all the planets is equal to a constant. This constant being rather small, it follows that all the inclinations will remain small, except for a planet with very small mass. Leverrier afterwards computed that for Mercury the extremes for eccentricity and inclination are 0.226 and 9°.17.

The importance of these researches consisted in that they raised in science and in the minds of the scientists the problem of the future, the durability of the world. When in the 17th and 18th century continuous changes were discovered in the orbits of the planets and the moon, it was feared that they could in the long run destroy the structure of the system. When theory attributed them to the mutual attraction of the celestial bodies the question was asked whether the perturbations could increase to such amounts as to upset the prevailing order. Laplace here gave the first reassuring answer in demonstrating the stability of the solar system in regard to the mechanical effects of the mutual attraction. The impression it made upon contemporary minds can be found reflected in nearly all popular books on astronomy from the first half of the 19th century, often with reference to the wisdom of the Creator who, by having made all the planets revolve in the same direction and their orbits nearly circular and little inclined, had provided for the eternity of the world structure on which our life depends.

Laplace himself, however, had already pointed out the restriction in this demonstration; the simple expressions derived for the secular in-